Introduction to the Theory of Computing 1.

Exercise-set 12.

1. The matrix of the linear mapping $f : \mathbf{R}^4 \to \mathbf{R}^3$ is given below. Determine the values $\dim \mathrm{Im} f$ and $\dim \mathrm{Ker} f$ and find bases in the subspaces $\mathrm{Im} f$ and $\mathrm{Ker} f$, resp.

$$\begin{pmatrix} 1 & 3 & 1 & 5 \\ 4 & 12 & 1 & 14 \\ 3 & 9 & 1 & 11 \end{pmatrix}$$

2. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a linear mapping and let $A = [f]$ be the matrix of f. Decide whether the following statements are true or not:
a) If $\mathrm{Ker} f$ contains a vector different from the zero vector, then $\det A = 0$.
b) If $\det A = 0$ then $\mathrm{Ker} f$ contains a vector different from the zero vector.
c) If $\mathrm{Im} f \subseteq \mathrm{Ker} f$ then $A^2 = 0$.
d) If $A^2 = 0$ then $\mathrm{Im} f \subseteq \mathrm{Ker} f$.

3. Let $f, g : \mathbf{R}^2 \to \mathbf{R}^2$ be linear transformations. Determine the linear transformation $g \circ f$, if
(a) f is the reflection through the x axis and g is the reflection through the y axis;
(b) f is the projection onto the x axis and g is the projection onto the y axis.

4. Determine the matrix of the linear transformation which rotates the vectors of the (3-dimensional) space around the z axis by 90 degrees and reflects them through the xy plane. Determine the image of the vector $\underline{i} + \underline{j} + \underline{k}$.

5. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation. Decide whether the following statements are true or not:
a) If f is invertible then $\mathrm{Ker} f = \{\underline{0}\}$.
b) If $\mathrm{Ker} f = \{\underline{0}\}$ then f is invertible.
c) If f is invertible then $\mathrm{Im} f = \mathbf{R}^n$.
d) If $\mathrm{Im} f = \mathbf{R}^n$ then f is invertible.

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6. MT'09 (a) Is there a linear transformation on $\mathbf{R}^3$ (the usual 3-dimensional vector space) whose image and kernel are the same?
(b) Is there a linear transformation on $\mathbf{R}^2$ whose image and kernel are the same?

7. MT'14 Let $f : \mathbf{R}^{20} \to \mathbf{R}^{10}$ be a linear mapping, $B = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_{20}\}$ a basis in $\mathbf{R}^{20}$, and $\underline{v} \in \mathbf{R}^{10}$, $\underline{v} \neq \underline{0}$ a fixed vector. Determine the value of $\dim(\mathrm{Ker} f)$ if we know that f maps each of the vectors $\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_{20}$ to the vector $\underline{v}$.

8. MT+'14 Let $f : \mathbf{R}^{10} \to \mathbf{R}^3$ be a linear mapping.
a) Show that there exist 7 linearly independent vectors in $\mathbf{R}^{10}$ in such a way that f maps them to the same vector.
b) Is the same statement true with 8 vectors instead of 7?

9. MT'15 Let A be a 6×9 matrix. We know that there are 5 linearly independent vectors, $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_5 \in \mathbf{R}^9$ for which $A \cdot \underline{v}_1 = \underline{0}$, $A \cdot \underline{v}_2 = \underline{0}, \ldots, A \cdot \underline{v}_5 = \underline{0}$ holds. Show that in this case $r(A) \leq 4$.

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10. For the linear transformation $f : \mathbf{R}^3 \to \mathbf{R}^3$ and the basis $B = \{\underline{b}_1 = (1,0,0)^T, \underline{b}_2 = (2,1,0)^T, \underline{b}_3 = (2,2,1)^T\}$ holds that $f(\underline{b}_1) = \underline{b}_2, f(\underline{b}_2) = \underline{b}_3$ and $f(\underline{b}_3) = \underline{b}_1$. Determine the matrices $[f]_B$ and $[f]$.

11. The linear transformation $f : \mathbf{R}^2 \to \mathbf{R}^2$ maps the vector $\underline{b}_1 = (2,3)^T$ to $(2,5)^T$ and the vector $\underline{b}_2 = (0,2)^T$ to $(2,1)^T$.
a) Determine the matrix $[f]_B$, i.e. the matrix of f in the basis $B = \{\underline{b}_1, \underline{b}_2\}$.
b) Determine $[f]$, the matrix of f.
c) What is the image of the vector $(4,6)^T$?

12. MT'10 Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation, and $B = \{\underline{b}_1, \underline{b}_2\}$ a basis of $\mathbf{R}^2$. Suppose that the matrix $[f]_B$ (in the basis B) is the matrix A below. Determine the value of p and q if we know that $f(\underline{b}_1) = \underline{b}_2$.

$$A = \begin{pmatrix} p & \sqrt{2} \\ q & \sqrt{3} \end{pmatrix}$$