Toughness and Edge–toughness

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Abstract

We generalize the definition of toughness and define a new property of graphs, the *egde–toughness*. In this new definition edges are also allowed to be deleted besides vertices. A number of results are given on the relation of toughness and edge–toughness.

1 Introduction

In this paper only finite, undirected and simple graphs are considered. In 1973 Chvátal [4] introduced the definition of toughness. Many authors investigated the relation of toughness and hamiltonicity since Chvátal's paper. A good survey of the topic is [2]. Let $\omega(G)$ denote the number of components of a graph G. If $T \subseteq V(G)$ then the graph $G-T$ is defined as follows. $V(G-T) = V(G)-T$ and $(u,v) = e \in E(G-T)$ iff $e \in E(G)$ but non of u and v is in T. A graph G is *t–tough* if $|S| \geq t\omega(G-S)$ for every subset S of the vertex set $V(G)$ with $\omega(G-S) > 1$. The *toughness* of G, denoted by $t(G)$, is the maximum value of t for which G is t-tough (taking $t(K_n) = \infty$ for all $n \geq 1$). A *k–factor* of a graph is a k–regular spanning subgraph. It is worth mentioning that a t–tough graph is always $2t$-vertex-connected.

Chvátal [4] investigated the relation of toughness and hamiltonicity. Being 1–tough is clearly a necessary condition for a graph to be hamiltonian. He conjectured that there exists a finite constant t_0 such that every t_0–tough graph is hamiltonian. This conjecture is still open. Enomoto, Jackson, Katerinis and Saito in [5] showed that if there is such a t_0 then $t_0 \geq 2$. In fact it is a

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consequence of a more general conjecture of Chvátal which was proved in their paper.

Theorem [Enomoto at al.] *Let $k \geq 1$. For any $\varepsilon > 0$ there exists a $(k-\varepsilon)$–tough graph G on v vertices with $v \geq k+1$ and kv even which has no k–factor.*

For $k = 2$ this means that $(2-\varepsilon)$–tough graphs were shown which have no 2-factor and therefore they are not hamiltonian. Hence if there exists a t_0 then $t_0 \geq 2$. The minimum degree of the graph in [5] in the case $k = 2$ is 4. In [3] the authors ask if there are graphs with larger minimum degree. At the time of writing this paper the author was not aware that Bauer, Broersma, van den Heuvel and Veldman found such graphs to be published in a forthcoming paper [1].

If we are trying to show that $t_0 > 2$, then we must show that there is a 2–tough graph which is not hamiltonian. How can we prove that the graph is not hamiltonian? The result of Enomoto at al. shows that proving that there is no 2–factor will not work. Therefore we have to find some other method to prove non–hamiltonicity.

Another method would be showing non–path–toughness [8]. Let $l(G)$ be the minimum number of vertex disjoint paths that are needed to cover all the vertices of G and $r(G) = \max\{l(G-X) - |X| | X \subseteq V(G) \text{ and } l(G-X) \neq 1\}$. A graph G is *path–tough* iff $r(G) \leq 0$. But unfortunately there is no general method for showing non–path–toughness.

Mader in [7] and later Lomonosov in [6] investigated the question of cycles through prescribed elements of a graph. A special case of this problem is of course the hamiltonian cycle problem where the set of prescribed elements of the graph is the set of all vertices. Some definitions and a Lemma will be presented here from this paper. The general form is not needed, therefore only a special case is given here.

In general the A–cycle problem is to decide whether there exists a cycle containing all elements of A, where A is a collection of vertices and edges from the graph. But let us restrict ourselves to the case when $A \subseteq V(G)$. For sake of simplicity we will suppose that the elements of A are pairwise non-adjacent. Otherwise a 2-degree vertex is put into the middle of the edge connecting the two vertices. (This new vertex is not in A in this case.)

We say that a pair (X,Y), where $X \subseteq V(G-A)$ and $Y \subseteq E(G-A-X)$, *disconnects* A, if $G-X-Y$ has no path connecting two vertices of A. ($G-X-Y$ is obtained by deleting the vertices of X and the edges of Y from G, but the endvertices of the edges in Y are in the graph $G-X-Y$.) Let $Q_1, Q_2, \ldots, Q_n$ be the components of the edge–induced subgraph $G(Y)$. The vertices of $G(Y)$ are the vertices of G which are incident to an edge in Y

Fig. 1. Examples of A-separators

and the edges of $G(Y)$ are those edges of G whose endvertices are both in the vertex set of $G(Y)$. Therefore there may be edges in $G(Y)$ which are not in Y. Let H be a graph and B a subgraph of H, then the boundary of B in H is defined by $\mathrm{bd}_H(B) = \{v \in V(B) | v \text{ has a neighbour outside of } B\}$ and $\mathrm{in}_H(B) = V(B) - \mathrm{bd}_H(G)$. Therefore in our case

$$\mathrm{bd}_{G-X}(Q_i) = \{v \in V(Q_i) \mid v \text{ has a neighbour outside of } Q_i \text{ and } X\}$$

and

$$\mathrm{in}_{G-X}(Q_i) = V(Q_i) - \mathrm{bd}_{G-X}(Q_i).$$

The following lemma may be formulated.

Lemma 1 (see [7]) *If there exists a pair (X, Y) disconnecting A, such that*

$$|A| > s(X, Y; G) := |X| + \sum_{i=1}^{n} \left\lfloor \frac{|\mathrm{bd}_{G-X}(Q_i)|}{2} \right\rfloor, \tag{1}$$

(where $\lfloor x \rfloor$ denotes the largest integer satisfying $\leq x$,) then there is no cycle containing all vertices of A.

A pair (X, Y) which satisfies 1 is called an *A–separator*. Figure 1 presents some examples of A–separators.

Lomonosov investigates A-separators in [6], our aim in the present paper is different, we are interested in hamiltonian cycles.

Let us call a path in a graph G an AA–path iff its endvertices are different vertices in A, and let $m(A, G)$ denote the number of all disjoint AA–paths in G. For a pair (X, Y) being an A–separator in G, each AA–path in G must contain a vertex from X or an edge from Y, in the latter case therefore two vertices of some Q_i. But only $\left\lfloor \frac{1}{2} |\mathrm{bd}_{G-X}(Q_i)| \right\rfloor$ disjoint AA–paths can cross the component Q_i, thus

$$m(A, G) \leq s(X, Y; G). \tag{2}$$

Considering (1) this implies that there is no cycle in G containing all vertices of A, which proves Lemma 1.

If (X, Y) is an A–separator in G then obviously, $(\emptyset, Y)$ is an A–separator in the graph $G - X$ with $s(\emptyset, Y; G - X) = s(X, Y; G) - |X|$, and we get analogously

$$m(A, G - X) \leq s(\emptyset, Y; G - X) = s(X, Y; G) - |X|. \tag{3}$$

This implies that $l(G-X) > |X|$, since if one could cover $G-X$ (including all vertices of A) with $r \leq |X|$ vertex disjoint paths then there must be at least $|A| - r \geq |A| - |X|$ disjoint AA–paths in $G-X$, i. e. $m(A, G-X) + |X| \geq |A|$, in contradiction to (3) and (1). This proves the following lemma.

Lemma 2 *If there exists an A-separator in G then G is not path–tough.*

Note that presenting an A-separator is an easy way of proving non–path–toughness in a graph.

2 t–edge–toughness

Obviously, if there is a subset of vertices A such that there exists an A–separator then there is certainly no hamiltonian cycle in the graph. This is a new method to prove non–hamiltonicity. It is a generalization of the previous method where non-hamiltonicity is proved with proving non-1-toughness. The "proof" for non-hamiltonicity is the cut set in that case, while the "proof" in our case is a "cut set" consisting of vertices and edges. However the set A is not really needed for this, therefore we give a new definition here.

Definition 1 *G is t–**edge–tough** if G is connected and*

$$\omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) \leq \frac{s(X, Y; G)}{t}$$

holds for every $X \subseteq V(G)$ and $Y \subseteq E(G - X)$ satisfying

$$\omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) > 1,$$

using the notations introduced above. The **edge–toughness** *of G, denoted* *by $t_e(G)$, is the maximum value of t_e for which G is t_e–edge–tough (taking $t_e(K_n) = \infty$ for all $n \geq 1$).*

Two observations are immediate.

Observation 1 *If G is t–edge–tough then it is t–tough.*

PROOF. The assertion follows from the definition above taking $Y = \emptyset$. $\square$

Observation 2 *If G is hamiltonian then it is 1–edge-tough.*

PROOF. It is enough to show that if G is not 1–edge–tough then there is a vertex set A and a pair (X, Y), such that this pair is an A–separator. Obviously if G is not 1–edge–tough then there exists a pair (X, Y), such that

$$\omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) > s(X, Y; G).$$

It will be proved that every component of $G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)$ contains at least one vertex which is disjoint from each $\mathrm{bd}_{G-X}(Q_i)$. Suppose that there is a component which contradicts our claim. All vertices in this component are in one of the vertex sets $\mathrm{bd}_{G-X}(Q_i)$. It is easy to see that there is no edge connecting a vertex of Q_i to a vertex of Q_j where $i \neq j$, since Q_i and Q_j are different components in $G(Y)$. If v is a vertex of our component and $v \in \mathrm{bd}_{G-X}(Q_p)$ then v must have a neighbour outside of Q_p and X by definition and, as we have seen, this neighbour cannot be in $\mathrm{bd}_{G-X}(Q_j)$ for any j, either. Hence this neighbour is disjoint from each $\mathrm{bd}_{G-X}(Q_i)$, a contradiction.

To construct A chose one vertex from each component of $G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)$ such that each of them is disjoint from each $\mathrm{bd}_{G-X}(Q_i)$. All edges in Y are edges of some Q_i therefore the vertices of A are not incident to any edge in Y. This implies that the elements of A are pairwise nonadjacent. On the other hand it is easy to see that (X, Y) is an A-separator. $\square$

3 Connection between toughness and edge–toughness

It may be possible to construct a non–1–edge–tough, but 2–tough graph using Observation 2. Such a graph would show that if there exists a t_0 in Chvátal's conjecture then $t_0 > 2$.

But this is not the case, the following theorem will be proved, instead.

Theorem 1 *If G is $2t$–tough then it is t–edge–tough.*

PROOF. For any $X \subseteq V(G)$ and any $Y \subseteq E(G-X)$ satisfying

$$\omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) > 1,$$

let S consist of all vertices of X and for every $i = 1, \ldots, n$ of $2\left\lfloor \frac{1}{2}|\mathrm{bd}_{G-X}(Q_i)|\right\rfloor$ vertices of $\mathrm{bd}_{G-X}(Q_i)$. Obviously

$$|S| = |X| + 2\sum_{i=1}^{n}\left\lfloor \frac{|\mathrm{bd}_{G-X}(Q_i)|}{2}\right\rfloor = 2s(X, Y; G) - |X|.$$

It will be shown now that

$$\omega(G-S) \geq \omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right)$$

holds. A set A is constructed containing exactly one vertex from each component of $G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)$ and containing no vertex from any $\mathrm{bd}_{G-X}(Q_i)$. It was shown in the proof of Observation 2 that every component contains such a vertex. It is easy to see that A satisfies $A \cap S = \emptyset$. In this way all vertices of A are vertices of $G - S$. It will be shown that the vertices of A are contained in different components of $G - S$, which proves our claim. Suppose there is a path in $G - S$ connecting two vertices of A. Since (X, Y) disconnects A this path must contain a vertex from X or an edge from Y. $X \subseteq S$, therefore the path cannot contain a vertex from X. Suppose now that the path contains an edge from Y and that this edge is an edge of Q_p. The first and the last edge of the path cannot be in Y, and therefore in Q_p, since A is not incident to any edges from Y by our construction. This implies that there must be at least two inner vertices of this path each of which is incident to an edge from Q_p and to an edge which is not in Q_p. This implies that these two vertices must be elements of $\mathrm{bd}_{G-X}(Q_p)$. By our construction of S, $G - S$ contains at most one of these two vertices, therefore there is no path connecting two vertices of A.

Now because G is $2t$–tough we get

$$\begin{aligned} \omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) &\leq \omega(G-S) \leq \\ &\leq \frac{|S|}{2t} = \frac{2s(X,Y;G) - |X|}{2t} \leq \frac{s(X,Y;G)}{t}, \end{aligned}$$

thus G is t–edge-tough. $\square$

Fig. 2. The construction with $\delta = 4$ and $q = 2$.

We conjecture that the converse of Theorem 1 is true for no t.

Conjecture 1 *For every $t > 0$ and $\varepsilon > 0$ there exists a graph G such that G is t–edge–tough and $t(G) < t + \varepsilon$.*

For $t = 1$ such graph is a complete split graph $S_{n+1,n}$ defined as follows.

Let $\{a_1, \ldots, a_{n+1}, b_1, \ldots b_n\}$ be the set of vertices. Let $\{a_1, \ldots, a_{n+1}\}$ span a complete graph K_{n+1} and $\{b_1, \ldots, b_n\}$ a completely nonconnected graph $\overline{K_n}$. Connect a_i and b_j for all i, j. This graph is 1–edge–tough, because it contains a hamiltonian cycle (Observation 2). On the other hand $t(S_{n+1,n}) = \frac{n+1}{n}$.

It is possible that Theorem 1 is sharp, so the following conjecture can be stated.

Conjecture 2 *For every $\varepsilon > 0$ there exists a $(2t - \varepsilon)$–tough but non–t–edge–tough graph.*

We can prove this conjecture in the special case $t = 1$. In addition the graphs constructed below are new examples of $(2-\varepsilon)$–tough, non–hamiltonian graphs in which the minimum degree is an arbitrary integer $\delta \geq 4$.

Theorem 2 *For every $1 \geq \varepsilon > 0$ and every integer $\delta \geq 4$ there exists a $(2 - \varepsilon)$–tough but non–1–edge–tough graph with minimum degree δ.*

PROOF. Let q be an integer, such that $q \geq \frac{2}{\varepsilon}$. The vertex and edge sets of the graph G is given below. (See Figure 2)

$$\begin{aligned} V(G) = \{&a_1, \ldots, a_{(2q+1)(\delta-1)}, b_1, \ldots, b_{(2q+1)(\delta-1)}, \\ &\qquad c_1, \ldots, c_{(2q+1)(\delta-1)}, x_1, \ldots, x_{\delta-2}\}, \\ E_1 = \{&\{a_i, b_i\} \mid 1 \leq i \leq (2q+1)(\delta-1)\}, \end{aligned}$$

$$
\begin{aligned}
E_2 &= \{\{a_i, c_i\} \mid 1 \le i \le (2q+1)(\delta-1)\}\,, \\
B_1 &= \{\{b_i, b_j\} \mid 1 \le i, j \le 2q+1, i \ne j\}\,, \\
B_2 &= \{\{b_i, b_j\} \mid 2q+2 \le i, j \le 4q+2, i \ne j\}\,, \\
&\ \vdots \\
B_{(\delta-1)} &= \{\{b_i, b_j\} \mid (2q+1)(\delta-2)+1 \le i, j \le (2q+1)(\delta-1), i \ne j\}\,, \\
C_1 &= \{\{c_i, c_j\} \mid q+1 \le i, j \le 3q+1, i \ne j\}\,, \\
C_2 &= \{\{c_i, c_j\} \mid 3q+2 \le i, j \le 5q+2, i \ne j\}\,, \\
&\ \vdots \\
C_{(\delta-1)} &= \Big\{\{c_i, c_j\} \mid \Big(1 \le i \le q \text{ or} \\
&\qquad\qquad (2q+1)(\delta-1)-q \le i \le (2q+1)(\delta-1)\Big) \text{ and} \\
&\quad \Big(1 \le j \le q \text{ or } (2q+1)(\delta-1)-q \le j \le (2q+1)(\delta-1)\Big), i \ne j\Big\}, \\
X_1 &= \{\{x_1, a_i\}, \{x_1, b_i\}, \{x_1, c_i\} \mid 1 \le i \le (2q+1)(\delta-1)\}\,, \\
X_2 &= \{\{x_2, a_i\}, \{x_2, b_i\}, \{x_2, c_i\} \mid 1 \le i \le (2q+1)(\delta-1)\}\,, \\
&\ \vdots \\
X_{(\delta-2)} &= \Big\{\{x_{(\delta-2)}, a_i\}, \{x_{(\delta-2)}, b_i\}, \{x_{(\delta-2)}, c_i\} \mid 1 \le i \le (2q+1)(\delta-1)\Big\}\,, \\
E(G) &= E_1 \cup E_2 \cup_{k=1}^{\delta-1} B_k \cup_{k=1}^{\delta-1} C_k \cup_{k=1}^{\delta-2} X_k.
\end{aligned}
$$

Note that the edge sets $B_1, \dots, B_{(\delta-1)}$ and $C_1, \dots, C_{(\delta-1)}$ form $2(\delta-1)$ disjoint subgraphs K_{2q+1} and that $x_1, \dots x_{(\delta-2)}$ are connected to every other vertex.

It is easy to see that G is not 1–edge–tough. Let $X = \{x_1, \dots, x_{(\delta-2)}\}$ and $Y = B_1 \cup \cdots B_{(\delta-1)} \cup C_1 \cup \cdots \cup C_{(\delta-1)}$. In this way the subgraphs K_{2q+1} mentioned above are the components $Q_1, \dots, Q_{2(\delta-1)}$, each of the vertices b_i, c_i is in some $\mathrm{bd}_{G-X}(Q_j)$ and all $in_{G-X}(Q_j) = \emptyset$. So G is not 1–edge–tough since

$$
\begin{aligned}
\omega(G - X - Y) &= (2q+1)(\delta-1) > \\
&\qquad > (2q+1)(\delta-1) - 1 = \delta - 2 + 2(\delta-1)q = s(X, Y; G).
\end{aligned}
$$

Suppose that S is a cut set, such that $(2-\varepsilon)\omega(G-S) > |S|$. There may be two types of components in $G - S$. Let c_1 denote the number of components of the form $\{a_i\}$ and let c_2 be the number of the other components. The latter ones must contain some vertex b_i, c_i or x_i. Obviously $c_1 + c_2 = \omega(G-S)$.

If any of $x_1, \dots, x_{(\delta-2)}$ is not in S, than $G - S$ remains connected, therefore it may be supposed that $x_1, \dots, x_{(\delta-2)} \in S$.

If $c_2 = 0$ then there are no components containing any other vertex than a_i. This implies that all other vertices are in the cut set S. Hence

$$|S| \geq 2(2q+1)(\delta-1)+\delta-2 > (2-\varepsilon)((2q+1)(\delta-1)) \geq$$
$$\geq (2-\varepsilon)c_1 = (2-\varepsilon)\omega(G-S),$$

a contradiction.

The next case is $c_2 = 1$. If there are $c_1 \geq 0$ components of the first type, then S must contain all the neighbours of the vertices a_i forming these c_1 components. (Note that no two a_i's have a common neighbour except $x_1, \ldots, x_{\delta-2}$.) This gives at least $2c_1 + \delta - 2$ vertices for S. Therefore

$$|S| \geq 2c_1 + \delta - 2 > (2-\varepsilon)c_1 + (2-\varepsilon) =$$
$$= (2-\varepsilon)(c_1+1) = (2-\varepsilon)\omega(G-S),$$

a contradiction again.

Now all the other cases, namely $c_2 \geq 2$, will be proved. The nonempty intersections of S and the triples $\{a_i, b_i, c_i\}$ are divided into 2 types:

i) $S \cap \{a_i, b_i, c_i\} = \{b_i, c_i\}$,
ii) $S \cap \{a_i, b_i, c_i\} \neq \emptyset, \{b_i, c_i\}$.

One may draw $G - \{x_1, \ldots, x_{(\delta-2)}\}$ in a different way (see Figure 3).

The $2(\delta-1)$ complete graphs form a "cycle", such that there are q or $q+1$ parallel 2–paths connecting the adjacent complete graphs. Let us call a set of such parallel 2–paths a *tie*. There are $2(\delta-1)$ ties. A tie is said to be cut if S intersects each 2–path in the tie. It is easy to see, that to create c_2 components, each containing at least one vertex from one of the complete graphs, at least c_2 ties must be cut. Hence S must intersect at least $c_2 q$ triples. Each of these intersections may be of type i) or ii). The number of intersections of type i) must be c_1 by the definition of c_1. Let m denote the number of intersections of type ii). Using these notations we obtain $c_2 \leq \frac{c_1+m}{q}$ and $|S| \geq 2c_1 + m$. Therefore

$$\omega(G-S) = c_1 + c_2 \leq c_1 + \frac{c_1+m}{q}$$

holds. $q \geq \frac{2}{\varepsilon} \geq 2$ implies

$$(2-\varepsilon)\omega(G-S) \leq \left(2-\frac{2}{q}\right)\left(c_1+\frac{c_1+m}{q}\right) =$$
$$= 2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)c_1 + \frac{2}{q}\left(1-\frac{1}{q}\right)m \leq$$

Fig. 3. $G - \{x_1, \ldots, x_{(\delta-2)}\}$ drown differently

$$\leq 2\left(1 - \frac{1}{q^2}\right) c_1 + \frac{2}{q} m \leq 2c_1 + m \leq |S|,$$

which contradicts our assumption. The graph is $(2-\varepsilon)$-tough, indeed.

Finally one can check that in G each a_i vertex has δ neighbours, which is the minimum degree. $\square$

It is worth mentioning that Theorem 1 implies

$$\frac{t_e(G)}{t(G)} \geq \frac{1}{2}$$

and Observation 1 implies

$$\frac{t_e(G)}{t(G)} \leq 1.$$

This suggests the following generalization of Conjecture 1 and 2.

Conjecture 3 *For every* $\frac{1}{2} \leq c \leq 1$ *there exists a graph* G *for which*

$$\frac{t_e(G)}{t(G)} = c$$

holds.

It is natural to make the following conjecture as a version of Chvátal's conjecture.

Conjecture 4 *There exists a* t_1 *such that every* t_1*–edge–tough graph is hamiltonian.*

If such a t_1 exists then Theorem 1 implies that every $2t_1$-tough graph is hamiltonian and therefore Chvátal's conjecture is true. On the other hand, if Chvátal's conjecture is true then Observation 1 implies that Conjecture 4 is true. Hence one may say that this conjecture is equivalent to Chvátal's conjecture.

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