Odd subgraphs and matchings

M. Kano

Department of Computer and Information Sciences
Ibaraki University, Hitachi 316-8511 Japan
e-mail: kano@cis.ibaraki.ac.jp

Gyula Y. Katona [1]

Department of Computer and Information Sciences
Budapest University of Technology and Economics
Budapest, Pob.: 91, 1521 Hungary
e-mail: kiskat@cs.bme.hu

Abstract

Let G be a graph and $f : V(G) \to \{1, 3, 5, \ldots\}$. Then a subgraph H of G is called a $(1, f)$-odd subgraph if $\deg_H(x) \in \{1, 3, \ldots, f(x)\}$ for all $x \in V(H)$. If $f(x) = 1$ for all $x \in V(G)$, then a $(1, f)$-odd subgraph is nothing but a matching. A $(1, f)$-odd subgraph H of G is said to be maximum if G has no $(1, f)$-odd subgraph K such that $|K| > |H|$. We show that $(1, f)$-odd subgraphs have some properties similar to those of matchings, in particular, we give a formula for the order of a maximum $(1, f)$-odd subgraph, which is similar to that for the order of a maximum matching.

Key words: Odd subgraph, Graph factor, Matching.

We consider finite graphs which have no loops and no multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $|G|$ the order of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. For two vertices x and y of G, we write xy or yx for an edge joining x to y. Let

$$f \ : \ V(G) \ \longrightarrow \ \{1,\ 3,\ 5,\ 7,\ \cdots\}$$

be an odd integer valued function defined on $V(G)$, where we allow $f(v) > \deg_G(v)$ for some vertices v, and f always denotes this function throughout this paper. Then a subgraph H of G is called a $(1, f)$-*odd subgraph* if $\deg_H(x) \in$

[1] Research supported by OTKA grants OTKA T 029772 and T 030059, and The Japan Society for the Promotion of Science

$\{1, 3, \dots, f(x)\}$ for all $x \in V(H)$. A spanning $(1, f)$-odd subgraph is called a *$(1, f)$-odd factor* of G. If $f(x) = 1$ for all $x \in V(G)$, then a $(1, f)$-odd subgraph is a matching, and a $(1, f)$-odd factor is a 1-factor (i.e., a perfect matching). Note that for convenience, we define a *matching* as a subgraph with all degrees one. A $(1, f)$-odd subgraph H of G is said to be *maximum* if G has no $(1, f)$-odd subgraph K with $|K| > |H|$. A subgraph with all degrees odd is called an *odd subgraph*, and a spanning odd subgraph is called an *odd factor*. In this paper, we shall show some results on $(1, f)$-odd subgraphs, which are generalizations of those on matchings. Then we can expect that some other results on matchings can be generalized to those on $(1, f)$-odd subgraphs. However, there exist some theorems on matchings that cannot be directly generalized. Such an example is given in Theorem 10. Some results on $(1, f)$-odd factors, which are generalizations of results on 1-factors, can be found in [4], [6] and [9].

A component of a graph is said to be *odd* or *even* according to the parity of its order. We denote by $o(G)$ the number of odd components of G. For two graphs H and K, the *join* $H + K$ denotes the graph with vertex set $V(H) \cup V(K)$ and edge set $E(H) \cup E(K) \cup \{xy \mid x \in V(H) \text{ and } y \in V(K)\}$. Let $H \triangle K$ denote the subgraph formed by the symmetric difference of the two edge sets, so $V(H \triangle K) = V(H) \cup V(K)$ and $E(H \triangle K) = (E(H) \cup E(K)) - (E(H) \cap E(K))$. Let R be a subgraph of a graph G and X a subset of $V(G)$. Then we say that *R covers X* if $V(R) \supseteq X$, and that *R avoids X* if $V(R) \cap X = \emptyset$. For subsets A and B of a set X, if A is a subset B, then $A - B$ denotes $A \setminus B$, in particular, $X - A$ denotes $X \setminus A$. Other notation and definitions not defined here can be found in [1] or [8].

A criterion for a graph to have a $(1, f)$-odd factor is given in the following theorem, which is a generalization of Tutte's 1-factor Theorem ([8] p.84).

Theorem 1 ([4]) *A graph G has a $(1, f)$-odd factor if and only if*

$$o(G - S) \le \sum_{x \in S} f(x) \quad \textit{for all} \ \ S \subset V(G) \ . \tag{1}$$

We first give a formula for the order of a maximum $(1, f)$-odd subgraph, which is similar to the following.

Theorem 2 (Berge [3]; [8] p.90) *The order of a maximum matching M of a graph G is given by*

$$|M| = |G| - \max_{S \subseteq V(G)} \{o(G - S) - |S|\}.$$

Theorem 3 *The order of a maximum $(1, f)$-odd subgraph H of a graph G is* *given by*

$$|H| = |G| - \max_{S \subseteq V(G)} \{o(G-S) - \sum_{x \in S} f(x)\}.$$

In order to prove the above theorem, we need the following lemma.

Lemma 4 ([7], p.54) *Let G be a connected graph. Then the following statements hold.*
(i) If $|G|$ is even, then G has an odd factor.
(ii) If $|G|$ is odd, then G has an odd subgraph of order $|G|-1$.

Proof. We give a short proof to (i). Let n be an odd integer such that $n \geq |G|-1$, and define the function f by $f(x) = n$ for all $x \in V(G)$. Then by Theorem 1, G has a $(1,f)$-odd factor, which is the required odd factor.

Statement (ii) is an easy consequence of (i) since G has a vertex v such that $G - \{v\}$ is connected. $\square$

Proof of Theorem 3. Let H be a maximum $(1,f)$-odd subgraph of G, and let $d := \max_{S \subseteq V(G)}\{o(G-S) - \sum_{x \in S} f(x)\}$. Then $d \geq 0$ as $o(G) \geq 0$, and $|G| + d$ is even since $|G| \equiv o(G-S) + |S| \equiv d \pmod 2$.

For every odd component C of $G-S$, if $V(C)$ is covered by H, then there exists at least one edge of H that joins C to S. Thus at least $o(G-S) - \sum_{x \in S} \deg_H(x)$ odd components of $G-S$ are not covered by H. This implies $|H| \leq |G| - d$.

We next prove the reverse inequality. Let $G' := G + K_d$ be the join of G and the complete graph K_d of order d, and define $f' : V(G') \to \{1, 3, \ldots\}$ by $f'(x) = f(x)$ for all $x \in V(G)$ and by $f'(x) = 1$ for all $x \in V(K_d)$. Then $o(G') = 0$ since $|G| + d$ is even. Let X be a non-empty subset of $V(G')$. If $V(K_d) \not\subseteq X$, then $o(G' - X) \leq 1 \leq \sum_{x \in X} f'(x)$. If $V(K_d) \subseteq X$, then by the definition of d, it follows that

$$o(G' - X) = o(G - X \cap V(G)) \leq d + \sum_{x \in X \cap V(G)} f(x) = \sum_{x \in X} f'(x).$$

Hence by Theorem 1, G' has a $(1,f')$-odd factor F'. Let $M := F' - V(K_d)$. Then M is a spanning subgraph of G and has at most d vertices of even degree, some of which may be isolated vertices of M. Therefore M has at most d odd components.

By applying (i) or (ii) of Lemma 4 to each component of M according to whether its order is even or odd, we obtain an odd subgraph H of M such that $|H| \geq |M| - d = |G| - d$. Since H is a $(1,f)$-odd subgraph of G, the proof is complete. $\square$

Let H be a $(1, f)$-odd subgraph of a graph G. Then H is said to be *maximal* if G has no $(1, f)$-odd subgraph H_1 such that $V(H)$ is a proper subset of $V(H_1)$. Recall that H is said to be *maximum* if G has no $(1, f)$-odd subgraph H_2 such that $|H_2| > |H|$. Moreover, if H has a cycle C, then $H - E(C)$ is also a $(1, f)$-odd subgraph with vertex set $V(H)$. By repeating this procedure, we can obtain a $(1, f)$-odd subgraph H' which is a forest and whose vertex set is $V(H)$.

For a subgraph K of a graph G and edge subsets $X \subset E(K)$ and $Y \subset E(G) - E(K)$, we denote by $K - X + Y$ the subgraph of G induced by $(E(K) - X) \cup Y$. A path in a graph G connecting two vertices x and y is denoted by $P(x, y)$ or $P_G(x, y)$.

We now show another property of $(1, f)$-odd subgraphs, which is a generalization of the following property of matchings.

Theorem 5 ([5]; [8] p.88) *Let G be a graph, and B and R be subsets of $V(G)$ such that $|B| < |R|$. If there exists a matching which covers B and one which covers R, then there exists a matching which covers B and at least one vertex of $R \setminus B$.*

Theorem 6 *Let G be a graph, and B and R be subsets of $V(G)$ such that $|B| < |R|$. If there exists a $(1, f)$-odd subgraph which covers B and one which covers R, then there exists a $(1, f)$-odd subgraph which covers B and at least one vertex of $R \setminus B$. In particular, every maximal $(1, f)$-odd subgraph is a maximum $(1, f)$-odd subgraph.*

Proof. Let H_B and H_R be $(1, f)$-odd subgraphs which cover B and R, respectively. We may assume that both H_B and H_R are forests. If H_B contains a vertex in $R \setminus B$, then H_B itself is the desired $(1, f)$-odd subgraph. Thus we may assume that H_B avoids $R \setminus B$ (i.e., $V(H_B) \cap (R \setminus B) = \emptyset$). If an edge e joins a vertex in $R \setminus B$ to a vertex in $V(G) - V(H_B)$, then $H_B + e$ is the desired $(1, f)$-odd subgraph. Hence we may assume that every neighbor of a vertex of $R \setminus B$ is in $V(H_B)$.

For convenience, we call the edges of H_B and H_R *blue* and *red edges*, respectively. Let $F := H_B \triangle H_R - I$, where I is the set of isolated vertices of $H_B \triangle H_R$. The red and blue degrees of a vertex v in F, which are the numbers of red and blue edges of F incident with v, are denoted by $\deg_{Fr}(v)$ and $\deg_{Fb}(v)$, respectively. Note that $\deg_{Fr}(v) < \deg_{H_R}(v)$ and $\deg_{Fb}(v) < \deg_{H_B}(v)$ if and only if there is an edge in $E(H_R) \cap E(H_B)$ incident with v. For $v \in V(F)$, let $m(v) = \max\{\deg_{Fr}(v), \deg_{Fb}(v)\}$. Since F has no isolated vertices, $m(v)$ is a positive integer.

We now construct a new graph F' from F in the following way. Corresponding to each vertex v of F, we define one vertex v' or $m(v)$ vertices $v'_1, v'_2, \ldots, v'_{m(v)}$

of F', and for each edge e of F, we define one edge e' with the same color as e of F' and add some new red or blue edges to F' as follows:

Case (i) If $v \notin V(H_R) \cap V(H_B)$ or $m(v) = 1$, then we define a vertex v' of F'.

Case (ii) If $v \in V(H_R) \cap V(H_B)$ and $m(v) \geq 2$, then in F' we define $m(v)$ independent vertices, $v'_1, v'_2, \ldots, v'_{m(v)}$, that is, we split v into $m(v)$ distinct vertices of F'.

In Case (i), if a blue or red edge e is incident with a vertex v in F, then let an edge e' with the same color as e be incident with v' in F'. In Case (ii), for every blue edge e incident with v in F, we pick an arbitrary vertex from $v'_1, v'_2, \ldots, v'_{m(v)}$ to be an endvertex of a blue edge e' in F' but we pick a different one for each blue edge. This is possible since $m(v) \geq \deg_{Fb}(v)$. We apply the same procedure for the red edges. If $m(v) = \deg_{Fb}(v) > \deg_{Fr}(v)$, then we add some new red edges. In this case some vertices from $v'_1, v'_2, \ldots, v'_{m(v)}$ are not covered by red edges. We claim that $m(v) - \deg_{Fr}(v)$ is even. If there are k edges incident with v in $E(H_B) \cap E(H_R)$, then

$$\deg_{Fb}(v) = \deg_{H_B}(v) - k \quad \text{and} \quad \deg_{Fr}(v) = \deg_{H_R}(v) - k,$$

which proves our claim since $\deg_{H_B}(v) - \deg_{H_R}(v)$ is even. Therefore we can cover the vertices of $v'_1, v'_2, \ldots, v'_{m(v)}$ not covered by red edges with a new set of independent red edges, namely with a red matching (see Figure 1). Then for every vertex of $v'_1, v'_2, \ldots, v'_{m(v)}$, exactly one red edge and one blue edge are incident with it. Similarly if $m(v) = \deg_{Fr}(v) > \deg_{Fb}(v)$, then an even number of vertices of $v'_1, v'_2, \ldots, v'_{m(v)}$ are not covered by the blue edges, so we cover these by a new blue matching.

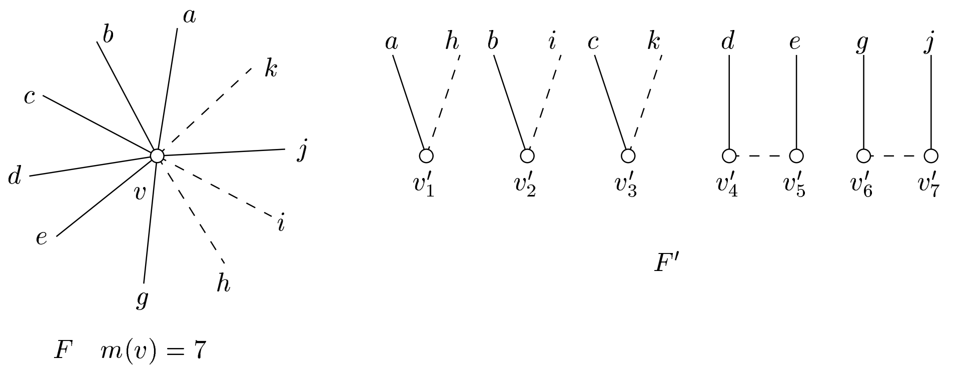

Fig. 1. A vertex $v \in V(H_R) \cap V(H_B)$ with $m(v) = 7$, and the vertices $v'_1, v'_2, \ldots, v'_{m(v)}$ of F'

Let B' and R' be the set of those vertices of F' which correspond to vertices of B and R, respectively. Also let F'_b and F'_r be the subgraphs induced by the blue and red edges of F', respectively. It is easy to see that every vertex in $V(F_b') \cap V(F_r')$ is incident with exactly one blue and one red edge, vertices of $V(F_b') \setminus V(F_r')$ are not incident with any red edges and vertices of $V(F_r') \setminus V(F_b')$ are not incident with any blue edges.

Recall that a *trail* is a walk such that all its edges are distinct. A trail connecting two vertices x and y is denoted by $T(x, y)$. A trail $T(x, y)$ of a graph G is said to be *maximal with respect to* y if $T(x, y)$ cannot be extended at y by adding a new edge of G to $T(x, y)$ (i.e., if $\deg_G(y) = \deg_{T(x,y)}(y)$).

Claim 7 *If there exists a path in F' with one endvertex in $R' \setminus V(F_b')$ and the other in $V(F_r') \setminus V(F_b')$ (such that these two endvertices are distinct), then there exists an $(1, f)$-odd subgraph in G which covers B and at least one vertex of $R \setminus B$.*

Proof. Suppose that there exists a path in F' between $R' \setminus V(F_b')$ and $V(F_r') \setminus V(F_b')$. Let $P(x', y')$ be such a shortest path connecting two distinct vertices $x' \in R' \setminus V(F_b')$ and $y' \in V(F_r') \setminus V(F_b')$. Then it follows immediately from the choice of $P(x', y')$ that $P(x', y')$ does not contain any vertex of $V(F_r') \setminus V(F_b')$ other than x' and y'.

We change the color of every edge in the path $P(x', y')$ from red to blue or from blue to red, and take the resulting blue subgraph, which is denoted by G_b'. The blue degree of x' and y' in G_b' became one. The blue degree of each vertex in $V(F_b') \setminus V(F_r')$ may be decreased by two, and thus still remains odd. The blue degree of each vertex in $V(F_b') \cap V(F_r')$ does not change.

Now define G_b to be the subgraph formed by those edges of F which correspond to edges of G_b' together with the edges in $E(H_B) \cap E(H_R)$, in particular, edges in G_b' joining two vertices of $v_1', v_2', \ldots, v_{m(v)}'$ give rise to no corresponding edge in G_b. We claim that G_b is the desired subgraph.

If a vertex v' is not in the path $P(x', y')$, then $\deg_{G_b}(v) = \deg_{H_B}(v) \le f(v)$ and it is odd. Clearly, $\deg_{G_b}(x) = 1 \le f(x)$ and $\deg_{G_b}(y) = 1 \le f(y)$, and both are odd. Suppose that v' is in $P(x', y') - \{x', y'\}$. If $v \in V(H_B) \setminus V(H_R)$, we have $\deg_{G_b}(v) = \deg_{Fb}(v) - 2 = \deg_{H_B}(v) - 2$. If $v \in V(H_B) \cap V(H_R)$ then $\deg_{G_b}(v) - \deg_{H_B}(v)$ is even, and so $\deg_{G_b}(v)$ is odd, however, it is possible that $\deg_{G_b}(v) > \deg_{H_B}(v)$. This means that the path $P(x', y')$ contains some new blue edges connecting two vertices of $v_1', v_2', \ldots, v_{m(v)}'$. If it contains k such edges then $\deg_{G_b}(v) \le 2k + \deg_{H_B}(v) \le \deg_{H_R}(v) \le f(v)$. Consequently, G_b is a $(1, f)$-odd subgraph, and covers $V(H_B)$ and $x \in R \setminus B$. $\Box$

Claim 8 *If there exists a trail $T(x', y')$ in F' connecting $x' \in R' \setminus V(F_b')$ and $y' \in V(F_b') \setminus (B' \cup V(F_r'))$ such that $T(x', y')$ is maximal with respect to y', then there exists an $(1, f)$-odd subgraph in G which covers B and at least one vertex of $R \setminus B$.*

Proof. Let $T(x', y')$ be a trail in F' connecting $x' \in R' \setminus V(F'_b)$ and $y' \in V(F'_b) \setminus (B' \cup V(F'_r))$ such that $T(x', y')$ is maximal with respect to y'.

Obviously, the degrees of x' and y' are odd in the trail, while the degree of any other vertex in the trail is even. Moreover, the red degree of x' is odd since its blue degree is zero and the blue degree of y' is odd and equal to $\deg_{F'_b}(y') = \deg_{H_B}(y')$ by the maximality of the trail.

If there exists a vertex $z' \in T(x', y')$ such that $z' \in V(F'_r) \setminus V(F'_b)$ and $z' \neq x'$, then there exists a path in F' connecting x' and z'. Thus by Claim 7, we can find the desired subgraph. Therefore we may assume that $T(x', y')$ does not contain any vertex of $V(F'_r) \setminus V(F'_b)$ other than x', and thus the red degree of each vertex of the trail, except x', is at most one.

We change the color of every edge in the trail $T(x', y')$ from red to blue or from blue to red, and take the resulting blue subgraph, which is denoted by G'_b. The blue degree of x' became odd and at most $\deg_{H_R}(x) \leq f(x)$, while the blue degree of y' is zero. The blue degree of any other vertex in $V(F'_b) \setminus V(F'_r)$ may be decreased by an even number, and thus still remains odd. The blue degree of vertices in $V(F'_b) \cap V(F'_r)$ does not change.

Now G_b is defined in the same way as in Claim 7. If a vertex v' is not in $T(x', y')$, then $\deg_{G_b}(v) = \deg_{H_B}(v) \leq f(v)$ and it is odd. We clearly have that $\deg_{G_b}(x) \leq f(x)$ and it is odd. On the other hand we have $\deg_{G_b}(y) = 0$, and so y is not covered by G_b, but since $y \notin B$ this does not cause any problem. Suppose that v' is in $T(x', y')$. Since for a vertex $v \in V(H_B) \setminus V(H_R)$, $\deg_{H_B}(v) - \deg_{G_b}(v)$ is even and non-negative, we have that $\deg_{G_b}(v) \leq \deg_{H_B}(v) \leq f(v)$ and $\deg_{G_b}(v)$ is odd. If $v \in V(H_B) \cap V(H_R)$ then $\deg_{H_B}(v) - \deg_{G_b}(v)$ is even but it may be negative. This means that the trail $T(x', y')$ contains some blue edges connecting two vertices of $v'_1, v'_2, \ldots, v'_{m(v)}$. If it contains k such edges then $\deg_{G_b}(v) \leq 2k + \deg_{H_B}(v) \leq \deg_{H_R}(v) \leq f(v)$. Consequently, G_b is a $(1, f)$-odd subgraph, and covers $H_B \setminus \{y\} \supseteq B$ and $x \in R \setminus B$. $\square$

Now we are ready to prove Theorem 6. We may assume that neither the conditions of Claim 7 nor those of Claim 8 hold. This means that for every trail $T(x', y')$ with $x' \in R' \setminus V(F'_b)$ which is maximal with respect to y', we have $y' \in B' \setminus V(F'_r)$ or $y' \in V(F'_b) \cap V(F'_r)$. Since the degree of any vertex in $V(F'_b) \cap V(F'_r)$ is exactly two, $y' \notin V(F'_b) \cap V(F'_r)$. Therefore for each vertex $x' \in R' \setminus V(F'_b)$, there exists a vertex $y' \in B' \setminus V(F'_r)$ such that there exists a $T(x', y')$ trail, which is maximal with respect to y'.

As $|B| < |R|$, we have $|B' \setminus V(F'_r)| \leq |B \setminus R| < |R \setminus B| = |R' \setminus V(F'_b)|$ since $V(F'_b) \cap (R' \setminus B') = \emptyset$. So there must be two distinct vertices $x', z' \in R' \setminus V(F'_b)$ and a vertex $y' \in B' \setminus V(F'_r)$ such that two trails $T(x', y')$ and $T(z', y')$ exist. Thus there exists a trail $T(x', z')$ and hence a path between x' and z', which is a contradiction. □

We state a conjecture, which is known to be true for matchings ([8] p.88).

Conjecture 9 *Let G be a graph, and B and R be subsets of $V(G)$ such that $|B| < |R|$. Then if there exists a maximum $(1, f)$-odd subgraph which avoids B and one which avoids R, then there exists a maximum $(1, f)$-odd subgraph which avoids B and at least one vertex of $R \setminus B$.*

We conclude this paper by stating a property which matchings possess but $(1, f)$-odd subgraphs do not.

Theorem 10 ([1] p.57) *Let G be a graph and W a subset of $V(G)$. Then G has a matching which covers W if and only if*

$$o(G - S \mid W) \leq |S| \quad \textit{for all} \quad S \subseteq V(G), \tag{2}$$

where $o(G - S \mid W)$ denotes the number of those odd components of $G - S$ whose vertices are contained in W.

Let G be the graph given in Figure 2, whose vertex set is $\{a, b, c, d, e, u, x, y, z\}$. Define the function f by $f(a) = f(b) = f(c) = f(d) = f(e) = f(u) = 1$ and $f(x) = f(y) = f(z) = 3$. Then for a subset $W = \{a, b, c, d, e, u\}$, G and W satisfies $o(G - S \mid W) \leq \sum_{x \in S} f(x)$ for all $S \subset V(G)$, but G has no $(1, f)$-odd subgraph which covers W. Hence $(1, f)$-odd subgraphs do not have exactly the same property as the one given in Theorem 10 for matchings.

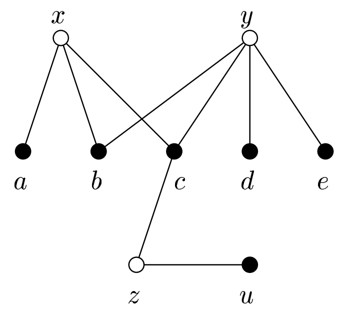

Fig. 2. A graph G having no $(1, f)$-odd subgraph covering W

Acknowledgment The authors would like to thank Akira Saito and Teng Cong for their helpful suggestions and comments. They also thank the referees for their usefull suggestions and comments.

This research is partly supported by Grant in Aid for Scientific Research of the Ministry of Education of Japan and the Japan Society for Promotion of Science.

References

[1] B. Bollobás, Extremal Graph Theory, Academic Press, (1978).

[2] C. Berge, Two theorems in graph theory, Proc. Nat. Acad. Sci. U.S.A. **43**, (1957) 842-844.

[3] C. Berge, Sur le couplage maximum d'un graphe, *C.R. Acad. Sci. Paris Ser.I Math.* **247**, (1958) 258-259.

[4] Y. Cui and M. Kano, Some results on odd factors of graphs, J. Graph Theory **12** (1988) 327-333.

[5] J. Edmonds and D. R. Fulkerson, Transversals and matroid partition, J. Res. Nat. Bur. Standards Sect. B **69B**, (1965) 14-153.

[6] M. Kano and H. Matsuda, Some results on $(1, f)$-odd factors, Combinatorics, Graph Theory and Algorithms , New Issues Press, **II** (1999) 527-533.

[7] L. Lovász, *Combinatorial Problems and Exercises*, North-Holland, (1979).

[8] L. Lovász and M. Plummer, *Matching Theory*, Annals of Discrete math. **29**, North-Holland, (1986).

[9] J. Topp and P. D. Vestergaard, Odd factors of a graph, Graphs and Combinatorics, **9** (1993) 371-381.