Packing paths of length at least two

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Keywords: Path factor, graph factor, packing

Abstract

We give a simple proof for Kaneko's theorem which gives a sufficient and necessary condition for the existence of a $\{P_3, P_4, P_5\}$-factor in a graph. Moreover we generalize this theorem and give a formula for the order of a maximum $\{P_3, P_4, P_5\}$-packing.

1 Introduction

We consider finite graphs without multiple edges and loops. Let P_n denote the path which contains n vertices and $n-1$ edges. For a subset X of vertices of graph G, $\langle X \rangle_G$ denotes the subgraph of G induced by X.

For a set $\{A, B, C, \ldots\}$ of connected graphs, a subgraph F of a graph G is called an $\{A, B, C, \ldots\}$-*packing* of G if each component of F is isomorphic to one of $\{A, B, C, \ldots\}$. An $\{A, B, C, \ldots\}$-packing is said to be maximum iff it covers a maximum number of vertices of G. If F is a spanning subgraph, then it is called a

*Research supported by OTKA grants OTKA T 029772 and T 030059.

perfect $\{A, B, C, \ldots\}$-packing or an $\{A, B, C, \ldots\}$-*factor*. Observe that a graph has a $\{P_3, P_4, P_5\}$-factor if and only if it has a $\{P_n \mid n \geq 3\}$-factor, which we abbreviate as $\{P_{\geq 3}\}$-factor. We will use this evidence throughout the paper.

A graph H is said to be *factor-critical* if $H - \{v\}$ has a 1-factor for all $v \in V(H)$. For a factor-critical graph H with $V(H) = \{v_1, v_2, \ldots, v_n\}$, add new vertices $\{u_1, u_2, \ldots, u_n\}$ together with new edges $\{v_i u_i \mid 1 \leq i \leq n\}$ to H. Then the resulting graph is called a *sun*. Note that K_2 is a sun and by definition, we regard K_1 also as a sun (see Figure 1). We call a sun with one vertex a *small sun*, otherwise a *big sun*. We denote by $\mathrm{Sun}(G)$ the set of sun components of G and let $\mathrm{sun}(G) = |\mathrm{Sun}(G)|$ the number of sun components.

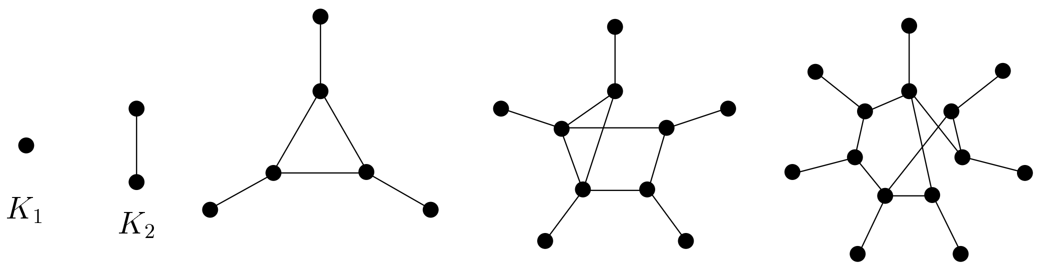

Figure 1: Suns

A vertex of degree one is called a *pendant vertex*, and an edge incident with a pendant vertex is called a *pendant edge*. For a vertex v of a graph G, we denote by $\deg_G(v)$ the *degree* of v in G, and by $N_G(v)$ the *neighborhood* of v in G. For a subset $S \subseteq V(G)$, we define $N_G(S) := \cup_{x \in S} N_G(x)$.

H. Wang [10] characterized the *bipartite* graphs having a $\{P_{\geq 3}\}$-factor. A. Kaneko recently generalized this theorem to general graphs. There are many results on component factors (for example, see [1] and [6]), but besides the well known theorem of Tutte [9] about f-factors and the more general theorem of Lovász [7] about (g, f)-factors all previous positive results (ie. that gives a good characterization) allows P_2 as a component. Hell and Kirkpatrick [4] proved that if H is a connected graph on at least 3 vertices then deciding whether a given graph G contains an $\{H\}$-factor is NP-complete. Thus, for example, we do not have a good characterization of graphs having a $\{P_3\}$-factor.

On the other hand we should mention the corresponding theorems of Hartvigsen (see [2] and [3]) about cycle-factors without short cycles.

Theorem 1 (Kaneko[5]) *A graph G has a $\{P_{\geq 3}\}$-factor if and only if*

$$\mathrm{sun}(G - S) \leq 2|S| \qquad \textit{for all} \quad S \subset V(G). \tag{1}$$

2 A Simple Proof of Theorem 1

The following lemma is an easy consequence of Hall's theorem ([8], Theorem 1.1.3).

Lemma 2 *Let B be a bipartite graph with partite sets X and Y such that $|Y| = 2|X|$. B has a $\{P_3\}$-factor, ie. a factor H such that $\deg_H(x) = 2$ for all $x \in X$ and $\deg_H(y) = 1$ for all $y \in Y$ if and only if*

$$|N_B(S)| \geq 2|S| \quad \textit{for all} \quad S \subseteq X.$$

Important properties of suns are described in the following two lemmas.

Lemma 3 *Let D be a sun which is not a K_1, and let $u_i v_i$ be a pendant edge of D. Then $D - \{u_i, v_i\}$ has a $\{P_4\}$-factor.*

Lemma 4 *Let D be a sun on more than two vertices, and v' is a pendant vertex of D. $D - v'$ has a $\{P_4, P_5\}$-factor.*

Proof: Let $V(D) = \{v, x, y, z, \ldots\} \cup \{v', x', y', z', \ldots\}$, where y' is the pendant vertex connected to y etc. Let F be the factor-critical graph $\langle\{v, x, y, z, \ldots\}\rangle_D$. Choose a neighbor x of v in F. Now $F - x$ has a perfect matching M, with some $vy \in M$. Take the path $\{y', y, v, x, x'\}$ and the $\{P_4\}$-s extending the other edges of M by pendant ones. $\square$

Proof of Theorem 1: Since no sun component can have a $\{P_{\geq 3}\}$-factor, it is easy to show that if G has a $\{P_{\geq 3}\}$-factor, then (1) holds.

We now prove the sufficiency by induction on $||G|| = |E(G)|$. Suppose that G satisfies (1). By setting $S = \emptyset$ condition (1) implies that no component of G is a sun. We may assume that G is connected and $|G| \geq 3$. We consider some cases.

Case 1 There exists $\emptyset \neq S \subset V(G)$ such that $\mathrm{sun}(G - S) = 2|S|$.

Choose a subset S of $V(G)$ so that S is maximal among all the subsets S' satisfying $\mathrm{sun}(G - S') = 2|S'|$.

Let C be any non-sun component of $G - S$. Then for a subset $X \subset V(C)$, we have

$$2|S \cup X| \geq \mathrm{sun}(G - (S \cup X)) = \mathrm{sun}(G - S) + \mathrm{sun}(C - X) = 2|S| + \mathrm{sun}(C - X).$$

Thus $\mathrm{sun}(C - X) \leq 2|X|$. Hence C satisfies (1), and so C has a $\{P_{\geq 3}\}$-factor by induction.

We define the bipartite graph B with vertex set $S \cup \mathrm{Sun}(G - S)$ by contracting every sun-component into a single vertex and removing multiple edges and edges inside S. Now $|\mathrm{Sun}(G - S)| = 2|S|$, and we show that

$$|N_B(X)| \geq 2|X| \quad \text{for all} \quad X \subseteq S. \tag{2}$$

Suppose that $|N_B(Y)| < 2|Y|$ holds for some $Y \subseteq S$. Then $\mathrm{Sun}(G-(S\setminus Y)) \supseteq \mathrm{Sun}(G-S)\setminus N_B(Y)$ holds, and thus

$$\mathrm{sun}(G-(S\setminus Y)) \geq \mathrm{sun}(G-S) - |N_B(Y)| > 2|S| - 2|Y| = 2|S\setminus Y|,$$

is implied, which contradicts the assumption (1). Thus (2) holds.

Therefore, by Lemma 2 graph B has a factor H such that $\deg_H(x) = 2$ for all $x \in S$ and $\deg_H(C) = 1$ for all $C \in \mathrm{Sun}(G-S)$. By making use of this factor, by Lemma 3 we can obtain a $\{P_{\geq 3}\}$-factor of G (see Figure 2).

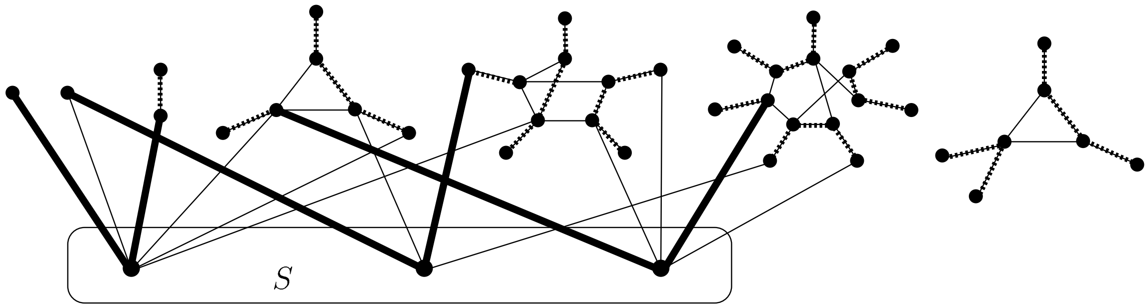

Figure 2: Extension of the $\{P_3\}$-factor of B to a $\{P_{\geq 3}\}$-factor in G

Case 2 $\mathrm{sun}(G-S) < 2|S|$ for all $\emptyset \neq S \subset V(G)$ and there exists $\emptyset \neq S' \subset V(G)$ for which $\mathrm{sun}(G-S') = 2|S'|-1$.

Choose a subset S so that S is maximal among all subsets S' satisfying $\mathrm{sun}(G-S') = 2|S'|-1$.

Let C be any non-sun component of $G-S$ and let $\emptyset \neq X \subset V(C)$. Using the maximality of S we obtain

$$2|S\cup X|-2 \geq \mathrm{sun}(G-(S\cup X)) = \mathrm{sun}(G-S)+\mathrm{sun}(C-X) = 2|S|-1+\mathrm{sun}(C-X).$$

$$\text{Thus}\quad \mathrm{sun}(C-X) \leq 2|X|-1. \tag{3}$$

Hence C has a $\{P_{\geq 3}\}$-factor by induction.

Claim 1 *If $G-S$ has a non-sun component then the desired $\{P_{\geq 3}\}$-factor exists.*

Proof: Let C be such a component, $v \in S$ and $w \in C$ such that vw is an edge. Let w^* be a new vertex and consider the graph $H := \langle C\rangle_G + ww^*$. Using (3) it is easy to see that H satisfies (1) for nonempty sets: $\mathrm{sun}(H-X) \leq \mathrm{sun}(C-X)+1 \leq 2|X|-1+1$. Clearly $||H|| < ||G||$, so by induction H has a $\{P_{\geq 3}\}$-factor containing a path P ending with $\{,w,w^*\}$ or H itself is a sun. In the latter case, by Lemma 3, H has a $\{P_2,P_4\}$-factor so that the only P_2 is $P = \{w,w^*\}$.

Construct bigraph B as above, but add a new pendant edge vw' to it. Now B satisfies (2) and we obtain a $\{P_3\}$-factor containing a path Q ending with $\{, v, w'\}$. Now take $P - w^*$ and $Q - w'$ and join them by the edge vw. Using this factor we can obtain a $\{P_{\geq 3}\}$-factor in the same way as in the previous case. $\square$

Claim 2 *If there exists $v \in S$ connected to zero, or more than one small suns in* $\mathrm{Sun}(G - S)$ *then the desired $\{P_{\geq 3}\}$-factor exists.*

PROOF: Construct B as before with the additional pendant edge vw'. Extend the $\{P_3\}$-factor of B as before to obtain a $\{P_{\geq 3}\}$-factor of $G + w'$ and delete now w'. The path containing v becomes a path of length one $\{v, w\}$ if $\{w\}$ is a small sun connected to v. By our assumptions another small sun w^* is also connected to v, and w^* is an endvertex of another path that can be extended by adding edge w^*v. $\square$

Claim 3 *If $\{w\} \in$* $\mathrm{Sun}(G - S)$ *a small sun and w is not pendant in G then the desired $\{P_{\geq 3}\}$-factor exists.*

PROOF: As w is not pendant, it is connected to some $v \in S$ and $v' \in S$, $v' \neq v$. Construct B as before with additional pendant edge vw'. We claim that $B' = B - vw$ satisfies (2). If $X \subset S$ then either $v \notin X$ and $N_{B'}(X) = N_B(X)$ or $v \in X$ and $|N_B(X)| \geq 2|X| + 1$ otherwise $S \setminus X$ would be a set with $\mathrm{sun}(G - (S \setminus X)) \geq 2|S \setminus X|$. For $X = S$ we need to prove $w \in N_{B'}(S)$ which is true because wv' is an edge. Now take the path ending in w' in the $\{P_{\geq 3}\}$-factor obtained using the $\{P_3\}$-factor of B', delete w' and connect the remains of this path to the path ending in w by edge vw. $\square$

So we may assume from now on that there are $|S|$ small suns in $\mathrm{Sun}(G - S)$ and $S - 1$ big suns. Each small sun is a pendant vertex of G and they are connected to different vertices in S.

Claim 4 *If every vertex of G with degree ≥ 2 has a pendant neighbor then the desired $\{P_{\geq 3}\}$-factor exists.*

PROOF: Let U be the set of vertices with degree ≥ 2. If $\langle U \rangle_G$ has a perfect matching, we are done. Otherwise there exists $X \subset U$ such that there are more than $|X|$ factor-critical components in $\langle U \setminus X \rangle_G$, consequently $\mathrm{sun}(G - X) > 2|X|$, which is a contradiction. $\square$

So we may assume there is a vertex with degree ≥ 2 which has no pendant neighbor in G. Clearly it is in a big sun $D \in \mathrm{Sun}(G - S)$. This means that $\langle D \rangle_G$

has a pendant vertex v' with neighbor $v \in D$ so that v' is connected (in G) to some $u \in S$ and if D is a K_2-component then v is also connected to S.

Subcase 2.1 $|D| = 2$

Let G' be the graph obtained from G by deleting the edge vv'. Now $\mathrm{sun}(G'-S) = 2|S|$. Construct B as in Case 1. It is easy to see that (2) is satisfied, so we obtain a $\{P_{\geq 3}\}$-factor as in Case 1.

Subcase 2.2 $|D| > 2$

Let the small sun neighbor of u be $D = \{w\}$. Construct B as before by adding pendant edge uw'. Take the $\{P_3\}$-factor of B, construct a $\{P_{\geq 3}\}$-factor of $G + w'$ and delete w'. Now we have a $\{P_{\geq 3}\}$-factor of G unless $\{w, u\}$ is a path of length one in this factor. If v' is an endvertex of another path of this factor then we can extend $\{w, u\}$ by adding edge uv'. Otherwise the path containing v' ends with $\{, s, v', v\}$ (by our construction this is the only possibility) where $s \in S, s \neq u$. Now delete edge $v'v$ from the factor as well as the edges inside D, add edge uv', and use Lemma 4 to obtain a $\{P_{\geq 3}\}$-factor of $D - v'$.

Case 3 $\mathrm{sun}(G - S) \leq 2|S| - 2$ for all $\emptyset \neq S \subset V(G)$.

If G has a pendant vertex u connected to v, then $\mathrm{sun}(G - \{v\} \geq 1)$, which contradicts the assumption of this case. Thus G is not a tree, and so we can find an edge e for which $G - e$ is connected. For every subset $\emptyset \neq S \subset V(G - e)$, we have

$$\mathrm{sun}((G - e) - S) \leq \mathrm{sun}(G - S) + 2 \leq 2|S| - 2 + 2 = 2|S|.$$

Moreover, $G - e$ is not a sun because having at least three vertices it would be a sun with at least three pendant vertices, but in this case G would have at least one pendant vertex as well. Therefore, $G - e$ has a $\{P_{\geq 3}\}$-factor by the inductive hypothesis, and so does G.

Consequently, the proof is complete. $\square$

3 The Order of a Maximum $\{P_{\geq 3}\}$-packing

Lemma 5 *Let B be a bipartite graph with bipartition $X \cup Y$ and $Y^* \subseteq Y$. Define*

$$\mathrm{def}(B) := \max_{Y' \subseteq Y}(|Y'| - 2|N_B(Y')|)$$

and

$$\mathrm{def}^*(B) := \max_{Y' \subseteq Y^*}(|Y'| - 2|N_B(Y')|).$$

If $\mathrm{def}(B) = |Y| - 2|X|$ then B has a $\{P_3\}$-packing which covers $|Y| - \mathrm{def}(B) = 2|X|$ vertices of Y including $|Y^*| - \mathrm{def}^*(B)$ vertices of Y^*, and covers all vertices in X.

PROOF: Define bipartite graph $B' = ((X \cup X') \cup Y, E')$ by adding for every vertex $x \in X$ a new vertex $x' \in X'$ and connecting x' to all neighbors of x. By Ore's theorem ([8], Theorem 1.3.1) there exist a matching M that covers $|Y| - \operatorname{def}(B') = |X \cup X'|$ vertices of Y and (consequently) all vertices of $X \cup X'$. Moreover, there exists another matching M^* that covers $|Y^*| - \operatorname{def}^*(B')$ vertices of Y^*. It is well known that this implies the existence of a matching which covers $X \cup X'$ and $|Y^*| - \operatorname{def}^*(B')$ vertices of Y^* (see [8], Exercise 1.4.3). This gives the desired $\{P_3\}$-packing in B if we contract all pairs x, x'. $\square$

Let $\mathrm{k}_2(H)$ denote the number of components of H which consist of an edge, in other terms the number of sun components isomorphic to a K_2.

Theorem 6 *The order of a maximum $\{P_{\geq 3}\}$-packing in a graph G is*

$$\operatorname{pp}(G) := |V(G)| - \max_{T \subseteq S \subset V(G)} (\operatorname{sun}(G-S) - 2|S| + \mathrm{k}_2(G-T) - 2|T|).$$

PROOF: It is proved first that the above expression is an upper bound on the order of a maximum $\{P_{\geq 3}\}$-packing. Let $T \subseteq S \subset V(G)$ such that $|V(G)| - \operatorname{pp}(G) = \operatorname{sun}(G-S) - 2|S| + \mathrm{k}_2(G-T) - 2|T|$. Clearly if F is a $\{P_{\geq 3}\}$-packing of G then there is a vertex in at least $\operatorname{sun}(G-S) - 2|S|$ components in $\operatorname{Sun}(G-S)$ which cannot be covered by F. Moreover, there are at least $\mathrm{k}_2(G-T) - 2|T|$ K_2-components of $G-T$ where none of the two vertices can be covered.

To prove the other direction choose $T \subseteq S \subset V(G)$ such that $\operatorname{sun}(G-S) - 2|S| + \mathrm{k}_2(G-T) - 2|T|$ is maximum.

Let C be a non-sun component of $G-S$. Then for a subset $X \subseteq V(C)$ we have

$$\operatorname{sun}(G-(S \cup X)) - 2|S \cup X| + \mathrm{k}_2(G-T) - 2|T| \leq \operatorname{sun}(G-S) - 2|S| + \mathrm{k}_2(G-T) - 2|T|$$

by the choice of S and T. Since $\operatorname{sun}(G-(S \cup X)) = \operatorname{sun}(G-S) + \operatorname{sun}(C-X)$ and $X \cap S = \emptyset$, it follows that $\operatorname{sun}(C-X) \leq 2|X|$ which implies that C has a $\{P_{\geq 3}\}$-factor. Hence, we may assume from now on that $G-S$ has only sun components.

Construct a bipartite graph B from G by contracting each sun component into a single vertex and removing multiple edges and edges inside S. The set of vertices which arose from the sun components in $G-S$ is denoted by Y, the set of vertices that arose from the contraction of K_2 components of $G-T$ is denoted by Q and the set of vertices which arose from the contraction of K_2 components of $G-S$ is denoted by Y^*. The K_2 components of $G-T$ are elements of $\operatorname{Sun}(G-S)$, because if there exists a K_2 component of $G-T$ such that $V(D) \cap (S-T) \neq \emptyset$ then we get a contradiction by considering $S \setminus V(D)$ and T. Thus $Q \subseteq Y^* \subseteq Y$ holds.

First we show that $\operatorname{def}^*(B) = \mathrm{k}_2(G-T) - 2|T|$. Suppose that

$$R - 2|N_B(R)| > \mathrm{k}_2(G-T) - 2|T|$$

holds for some $R \subseteq Y^*$. Then choosing $N_B(R)$ instead of T (and keeping S), obviously violates the choice of T and S.

Next we prove that $\operatorname{def}(B) = \operatorname{sun}(G - S) - 2|S|$. Suppose that

$$R - 2|N_B(R)| > \operatorname{sun}(G - S) - 2|S|$$

holds for some $R \subseteq Y$. Let $S' := N_B(R \cup Q)$ and $T' := N_B(R \cap Q)$. Since $N_B(Q) = T$, it is obvious that $S' = N_B(R) \cup T$ and $T' \subseteq N_B(R) \cap T$. In this way

$$\begin{gathered}\operatorname{sun}(G - S') - 2|S'| + \mathrm{k}_2(G - T') - 2|T'| \geq |R \cup Q| - 2|S'| + |R \cap Q| - 2|T'| = \\ |R| + |Q| - 2|S'| - 2|T'| \geq |R| + |Q| - 2|N_B(R) \cup T| - 2|N_B(R) \cap T| = \\ |R| + |Q| - 2|N_B(R)| - 2|T| > \operatorname{sun}(G - S) - 2|S| + \mathrm{k}_2(G - T) - 2|T|\end{gathered}$$

holds, which contradicts the choice of S and T.

By applying Lemma 5 a $\{P_3\}$-packing is obtained in B which covers all vertices in S, $|Y| - \operatorname{sun}(G - S) + 2|S|$ vertices of Y including $|Y^*| - \mathrm{k}_2(G - T) + 2|T|$ vertices of Y^*. (See Figure 3.)

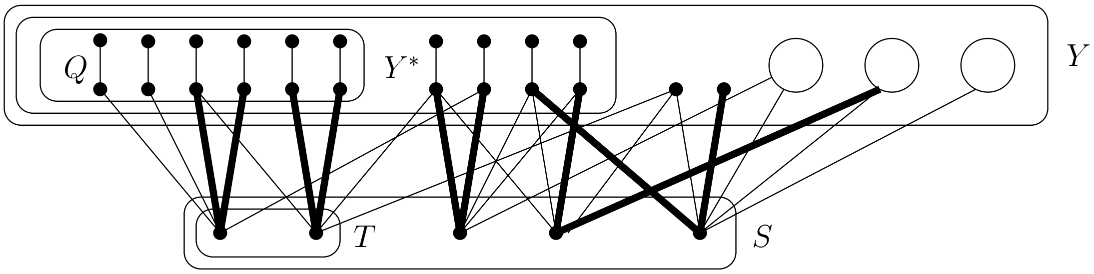

Figure 3: B with the desired factor.

In the original graph G we can extend this packing in the usual way by Lemma 3 (see Figure 2). On the other hand, the sun-components with more than two vertices which is not covered by the P_3-s in B can be almost covered by a $\{P_{\geq 3}\}$-packing by Lemma 4. This gives a $\{P_{\geq 3}\}$-packing of size $\operatorname{pp}(G)$. $\square$

Note, that

$$\max_{S \subset V(G)} (\operatorname{sun}(G - S) - 2|S|) = 0 \tag{4}$$

implies that $\mathrm{k}_2(G - T) - 2|T| \leq 0$ holds for any $T \subset V(G)$, because K_2 is a sun. This shows that

$$\max_{T \subseteq S \subset V(G)} (\operatorname{sun}(G - S) - 2|S| + \mathrm{k}_2(G - T) - 2|T|) = 0$$

holds if and only if (4) holds, hence Theorem 6 implies Theorem 1.

4 Acknowledgement

The authors are indebted to the organizers and sponsors of the 1st Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications (1999, Kyoto, Japan), where they learned the result of Kaneko and started to work on this paper.

References

[1] J. Akiyama and M. Kano, Factors and factorizations of graphs – A survey, *J. Graph Theory*, **9** (1985) 1-42.

[2] D. Hartvigsen, *Extensions of Matching Theory*, PhD. Thesis, Carnegie-Mellon University (1984).

[3] D. Hartvigsen, *The Square-Free 2-Factor Problem in Bipartite Graphs*, Integer Programming and Combinatorial Optimization, 7th IPCO Conference (Graz, 1999) LNCS 1610, 234-241.

[4] P. Hell and D. G. Kirkpatrick, *On the completeness of a generalized matching problem*, Proceedings of the Tenth Annual ACM Symposium on Theory of Computing (San Diego, 1978), 240-245.

[5] A. Kaneko, A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two, submitted.

[6] M. Loebl and S. Poljak, Subgraph packing – A survey, *Topics in Combinatorics and Graph Theory*, ed by R. Bodendiek and R. Henn, Physical-Verlag Heidelberg (1990).

[7] L. Lovász, *Subgraphs with prescribed valencies*, J. Combin. Theory **8** (1970), 391-416.

[8] L. Lovász and M. D. Plummer, *Matching Theory* (Annals of Discrete Math. 29), North-Holland (1986), Amsterdam.

[9] W. T. Tutte, *The factors of graphs*, Canad. J. Math. **4** (1952), 314-328.

[10] H. Wang, *Path factors of bipartite graphs*, J. Graph Theory **18** (1994), no. 2, 161-167.