Paths and Cycles in Graphs and Hypergraphs

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Contents

Introduction

The present thesis contains results from two well distinguishable topics. Chapters 1 and 2 are about Polyp Packing while Chapters 3–5 deal with different aspects of Hamiltonian graph theory. The reader finds notations, definitions and a brief introduction at the beginning of each Chapter except Chapter 2 for which Section 1.1 is relevant. Most of the terminology used in this thesis is standard and can be found in any text book on graph theory. The standard reference is [12] but we also mention [64] which is in Hungarian. All other notions are defined in the first Sections of the Chapters. In the present Chapter the author introduces some of the basic results from the two areas.

Polyp Packing

The Polyp Packing problem is a generalization of the Bin Packing problem which can be stated as follows.

BIN PACKING PROBLEM: *Given infinitely many bins of capacity K and a list L of t items with sizes $a_1, a_2, \ldots, a_t$ ($a_i \leq K$). Place the items into a minimum number of bins such that no bin contains items whose sum of sizes exceeds K.*

Although there are many algorithms which solves this problem, these require exponential number of steps of the size of the input. Moreover, we cannot expect a polynomial time algorithm because it is known that the problem is $\mathcal{NP}$–hard [46]. (In fact, the corresponding decision question is $\mathcal{NP}$–complete.) On the other hand, the Bin Packing problem models several practical problems in computer science and other areas. Some examples are the Cutting Stock problem [47], [48], the Assembly-line Balancing problem [25], [65], the Multiprocessor Scheduling problem [15], [21], [23], [49], [50], the File Allocation problem [42], [55], the Loading problem [33] and the Advertisement Scheduling problem [14]. Other applications can be found in [36], [83], [86].

Thus it seems to be important to find approximate, heuristic algorithms which hopefully compute "good" solutions sufficiently fast. From one point of view, it is difficult to compare these algorithms in practice, but one can analyze mathematically the performance of such algorithms to determine how close the constructed solutions approximate the optimal solutions. There are two widely used ways to measure the quality of these algorithms. One is the worst case performance ratio which we define below and the other one is the average performance ratio. Since this thesis deals with only the worst case performance, we do not define the average performance ratio here. The reader can find the definition and some results about average performance in [10], [24], [26], [27], [40], [52], [53].

Let A stand for a heuristic algorithm which solves the Bin Packing problem. The number of bins needed to pack a set L of items is denoted by $A(L)$, while the number of bins in an optimal packing is $OPT(L)$. Let us first define the worst case performance ratio for a fixed $OPT(L)$:

$$R_{A,i} = \max\left\{ \frac{A(L)}{OPT(L)} \,\middle|\, OPT(L) = i \right\}.$$

Now the *asymptotic performance ratio* of the worst case can be defined as

$$R_A = \limsup_{i\to\infty} R_{A,i}.$$

There is another way to classify the algorithms. An algorithm is *on-line* if it places the next item immediately without any information of the later coming items and no item is replaced after it has been placed. If we deal with such an algorithm then the items are given in an ordered list so L really denotes this list. The algorithms not satisfying these requirements are called *off-line*. It is clear that off-line algorithms have better performance, however, in some applications we are forced to use on-line algorithms. Another advantage of on-line algorithms is that they are usually fast and easily to executable.

One of the first and probably best know on-line algorithm is the *First Fit* algorithm.

FIRST FIT ALGORITHM [55]: *Let the bins be indexed as $B_1, B_2, \ldots$ with each initially filled to level zero. The items $a_1, a_2, \ldots, a_t$ will be placed in that order. To place a_i find the least indexed bin B_j for which*

$$\sum_{\text{items in } B_j} a_k \le K - a_i$$

and place a_i to B_j.

Johnson et al. had pioneering work in this area. They investigated the worst case performance of several other algorithms, too. The present thesis uses the methods they used in the proof of the following Theorem which implies $R_{\text{First Fit}} = 1.7$.

THEOREM [55] *For every t there exists a list L with $OPT(L) = t$ such that*

$$FF(L) > \frac{17}{10} OPT(L) - 8.$$

For every list L

$$FF(L) \le \frac{17}{10} OPT(L) + 2.$$

Since the present work does not consider other heuristic algorithms, instead of defining them we refer to [26], [38], [43], [55], [56], [70], [71], [82], [84] where the reader can find the most important algorithms and results.

In Chapter 1 a generalization of the Bin Packing problem, the Polyp Packing, is defined. Roughly speaking, the bins are replaced by a special graph class, the polyps, and the items by paths. A graph is called *p–polyp* if it consists of p simple paths of the same length and one endvertex of all these paths is a common vertex, the *center of the polyp*. It is proved that this problem is $\mathcal{NP}$–complete therefore it is important to investigate some approximating algorithms. We give a natural analogue of the First Fit algorithm for the Polyp Packing. In Chapter 1 an almost sharp upper and lower bound is given for the asymptotic performance ratio of the worst case of the edge disjoint Polyp Packing while Chapter 2 contains similar results for the vertex disjoint case. These results can be found in [59] and [60].

Some other versions of the Bin Packing problem are investigated in [2], [3], [16], [22], [26], [41], [43], [44], [45], [68], [69], [82].

Hamiltonian cycles

The Hamiltonian cycle problem is one of the first questions in graph theory. Euler "discovered" graph theory in 1736 [35]. In modern terminology he asked if it was possible to find a walk which crosses every edge of a given graph exactly once. In 1856 Hamilton invented a game, the "Icosian Game", consisting of a dodecahedron, each of its twenty vertices labeled with the name of a city. The object of the game was to make a "voyage around the world", visiting each city exactly once and returning to the initial point [51]. (Probably he was not thinking about attending conferences around the word.) In modern terminology: Is there a cycle in a given graph which contains each vertex exactly once?

Euler solved his problem himself and gave an easy necessary and sufficient condition. In contrast, the Hamiltonian cycle problem seems to be much harder to answer. Up to now, no easily verifiable necessary and sufficient condition is known. Moreover, we cannot expect such a condition since it is proved that this question is $\mathcal{NP}$–complete [46]. This gave rise to a growing number of conditions that are either necessary or sufficient for the existence of a Hamiltonian cycle. A wide variety of properties of graphs is used in the statements of these conditions. A good survey paper is [20].

There are only a few necessary conditions. The most important one is due to Chvátal [18] (see Section 4.1), every graph which contains a Hamiltonian cycle is 1–tough. Since the Hamiltonian cycle is a union of two 1–factors if n is even, it is a special 2-factor and it is a cycle which contains any given subset of vertices, any necessary conditions for these gives a necessary condition for the Hamiltonian cycle, too.

One of the earliest sufficient conditions was given by Dirac in 1952 (see Theorem 5.2 [30]) in which a lower bound on the degrees forces the existence of a Hamiltonian cycle. Several generalizations of his Theorem were found later [13], [17], [74], [76].

Is it possible to give weaker sufficient conditions if we assume that the graph is 1-tough (we exclude a trivial reason which forces non–Hamiltonicity)? There are some results in [4] and [57] which answer this question affirmatively, however, no big improvement has been achieved. Moreover, Chvátal thought that assuming high toughness really helps, he conjectured that there exists a t_0 such that every t_0–tough graph contains a Hamiltonian cycle. This conjecture is still open.

The main result in connection with this conjecture is due to Enomoto et al. [34] (see Theorems 4.1 and 4.2) which shows that $t_0 \geq 2$ if it exists. Some other papers related to this topic are [5], [6], [7], [39], [79], [86]. For survey papers we refer to [8], [9], [54].

Some other aspects of Hamiltonian graph theory are investigated in [1], [19], [66], [67], [80], [75].

Since there is no hope for an algorithm which finds a Hamiltonian cycle in polynomial time we are interested in "approximating Hamiltonian cycles". It is natural to consider a "long cycle" to be a good approximation. However, finding a long cycle is also a hard problem. In Chapter 3 of the present thesis a cycle is considered to be a good approximation if the distance of every vertex of the graph is relatively small from the cycle. The complexity of finding such a cycle is investigated there. The problem is $\mathcal{NP}$–compete if $0 < f(n) \leq n^{1-\varepsilon}$ while it is polinomially solvable if $f(n) \geq \frac{n}{c}$. The results appeared in [58]. It is worth mentioning that the definition of f–Hamiltonian cycle is very similar to the definition of D_λ–cycles in [4].

In the definition of toughness only vertices are deleted from the graph (see Section 4.1). In Chapter 4 we generalize this definition by allowing the deletion of edges besides vertices (Definition 4.6). It is easy to see that every Hamiltonian graph is 1–edge–tough. The connections of edge–toughness to toughness, Hamiltonicity and factors are investigated in Chapter 4. These results can be found in [61] and [62].

In Chapter 5 the analogue of the Hamiltonian cycle, the Hamiltonian chain, is defined in a new way (see Definition 5.4). A cyclic ordering of the vertex set is a *Hamiltonian chain* if any k consecutive vertex of this ordering forms a hyperedge of the given hypergraph. It seems that this definition has connections to other areas (see [29] and [37]). In Section 5.2 a Dirac type result is proved for 3–uniform hypergraphs. If any two distinct vertices are contained by sufficiently many hyperedges then the hypergraph contains a Hamiltonian chain (Theorem 5.7). An upper and a lower bound is given in Section 5.3 for the maximum number of edges of a hypergraph which does not contain a Hamiltonian chain. In Section 5.4 Hamiltonian nets are defined and a construction is given which shows that this notion differs from the Hamiltonian chain very much. These results can be found in [63].

Acknowledgment

First of all I would like to thank my mother and father who started my mathematical education in my very early ages. Since my father was drawing graphs during his work at home when I was about four years old, I learned the basics of graph theory very soon. His ongoing guidance was essential for me.

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Chapter 1

Edge Disjoint Polyp Packing

1.1 Introduction to Polyp Packing

Both of the following problems may be considered as a different formulation of the well known Bin Packing problem [55]: Given a simple path P, a set of simple paths possibly with many different lengths and a positive integer K. Is it possible to embed all the paths edge/vertex disjointly into K copies of P? (In the edge disjoint case the embedded paths may contain a vertex more than once, but each edge only once.)

G. O. H. Katona generalized this question: Given a graph G, a set of simple paths possibly with many different lengths and a positive integer K. Is it possible to embed all the paths edge/vertex disjointly into K copies of G?

It is easy to show that the vertex disjoint problem is equivalent to the Bin Packing problem if G has a Hamiltonian path and that the edge disjoint problem is equivalent to the Bin packing problem if G has an Eulerian path.

In Chapter 1 and 2 this question is investigated in a special case when the given graph is a polyp. A graph is called *p–polyp* if it consists of p simple paths of the same length and one endvertex of all these paths is a common vertex, the *center of the polyp*. The paths attached to the center are called *arms*. These problems are called *edge/vertex disjoint Polyp Packing problems*. Obviously, if the given graph is a p–polyp then the problem is called p–Polyp Packing. The main difference between the edge and vertex disjoint case is that several packed paths may contain the center of the polyp in the first case but only one in the second case (see Fig. 1.1). Note that there is no difference if $p = 3$.

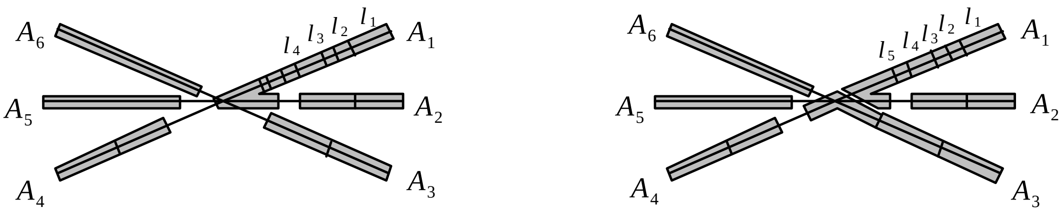

Fig. 1.1

In Chapter 1 first it is proved that both Polyp Packing problems are also $\mathcal{NP}$–complete like the original Bin Packing problem (see [46]). Then we define a variant of the First Fit algorithm and determine how many more polyps are needed in the worst case to pack the paths with the First Fit algorithm than with the optimal packing in the edge disjoint case. In

Chapter 2 the reader finds the corresponding results for the vertex disjoint case. Since there is no difference between the edge and vertex disjoint case if $p = 3$, the vertex disjoint case deals only with the case $p \geq 4$. Some of the proofs are similar to the proofs in Johnson et al. [55] which proves similar theorems about the original Bin Packing problem but it seems to be more difficult to apply the same techniques. Section 2.2 contains some concluding remarks concerning applications and open problems for both the edge and vertex disjoint problems.

1.2 $\mathcal{NP}$–completeness

THEOREM 1.1 *Both the edge and vertex disjoint p–Polyp Packing problems are $\mathcal{NP}$–complete for any given $p \geq 3$.*

PROOF: First of all it is trivial that both problems are in $\mathcal{NP}$, since one can check the validity of a given packing in polynomial time.

Edge Disjoint Case. We transform Bin Packing to p–Polyp Packing. If in the Bin Packing problem we consider only integer size items and a "large" integer size bin instead of rational size items, the problem remains $\mathcal{NP}$–complete, trivially. There is no real difference if odd integer size bins are not allowed.

So we really transform this restricted problem to the Polyp Packing problem. There are two cases depending on the parity of p. In the first case p is odd. Let the given set of integer sizes be $a_1, a_2, \ldots, a_t$, the size of the bin is $2s$ and the integer in the question is K. We create a set of paths with lengths $a_1, a_2, \ldots, a_t$ then add $\frac{p-1}{2}K$ copies of a $4s$ long path. So there are $t + \frac{p-1}{2}K$ paths. It is possible to pack the t items into K bins of size $2s$ if and only if one can pack these paths into K copies of a p–polyp with arm length $2s$.

The second case is the even p. A set of paths is created as before, paths with length $a_1, a_2, \ldots, a_t$ and additionally $\frac{p-2}{2}K$ paths of length $2s$ but the arms of the polyp now have length s.

If it is possible to pack the t items into K bins of size $2s$ then it is easy to see that one can pack the above set of paths into K p–polyps with arm length s. $\frac{p-2}{2}$ paths of size $2s$ are put into the first polyp and the paths corresponding the content of the first bin are put into the remaining space in the first polyp, etc.

On the other hand let us show that if one can pack the items into K p–polyps then there is a packing into K bins of size $2s$. Delete the arms containing one of the paths of the additional paths with size $2s$, clearly $2K$ arms remains. These arms contain all the paths which corresponds to the sizes $a_1, a_2, \ldots, a_t$. Now a matching is made between these arms. If there is path which has a part in two different arms, these arms are matched. The remaining arms of a polyp can be matched arbitrarily. It is clear that exactly K pairs are obtained in this way. So put the items corresponding to the contents of one pair to one bin.

Vertex Disjoint Case. Let the given set of integer sizes be $a_1, a_2, \ldots, a_t$, the size of the bin s and the integer in question K. A set of paths with lengths $a_1 - 1, a_2 - 1, \ldots, a_t - 1$ is created, K copies of a path with length s and $(p-2)K$ copies of a path with length $s - 1$ is added to this set.

First it is proved that if the items can be packed into K bins of size s then there is a packing of the above paths into K p–polyps with arm size s. Put one path of size s to the first polyp so that it covers one arm, thus the endvertex of it will cover the center of the polyp. Then we put $p-2$ paths of size $s-1$ to the other arms, the only way for this is when each of them covers all vertices of another arm. Finally the paths corresponding to the contents of the first bin will fit into the remaining arm of the first polyp vertex disjointly. Then this procedure is repeated with each polyp.

To complete the proof it must be shown that if there is a packing of the above paths into K p–polyps then we can pack the items to K bins of size s. It is clear that if there is a packing of the paths to the polyps then there is exactly one path of size s in every polyp. This path must cover the center. It is also possible that it covers vertices in two different arms, but the paths in these two arms contain at most s vertices. Therefore we may assume that it covers one arm completely and the center, and the other paths are in a separate arm. A path of size $s-1$ can be only in an arm alone. Thus the paths corresponding to the items are packed into K different arms. So the content of one arm goes simply to one bin and the proof is complete. ■

1.3 Edge Disjoint First Fit

Let us define the algorithm which will be investigated.

Let the p–polyps be indexed as $P_1, P_2, \ldots$, initially each empty. The arms of P_i are also indexed $A_{i,1}, \ldots, A_{i,p}$. The paths $l_1, l_2, \ldots, l_t$ will be placed in that order. To place l_i, we try to put it into $A_{1,1}$ so that it completely fits into it. If it is not possible we try to put it into $A_{1,2}$ and so on. If we find an arm which is not "full" we put l_i to this arm as far as possible from the center. If all the arms of P_1 are "full" we try to put l_i to $A_{1,1}$ and $A_{1,2}$ so that l_i will contain the center of P_1 and $A_{1,1}$ is filled completely. Then we try to put it into $A_{1,1}$ and $A_{1,3}$ and so on, we continue using the lexicographic order of the pair of arms. If l_i does not fit to P_1, we move on to P_2, etc.

This is called the First Fit algorithm. The number of polyps needed to pack a set L of paths is denoted by $FF_p(L)$, while the number of polyps in an optimal packing is $OPT_p(L)$. The set of paths may be viewed in the First Fit algorithm like a list since it gives us the order of the paths in the First Fit algorithm. So L really denotes this list while $|L|$ denotes the number of paths in the list. Let us first define the worst case performance ratio for a given number of polyps:

$$R_{p,i} = \max\left\{ \frac{FF_p(L)}{OPT_p(L)} \,\middle|\, OPT_p(L) = i \right\}.$$

Now the asymptotic performance ratio can be defined:

$$R_p = \limsup_{i \to \infty} R_{p,i}.$$

To make computations easier, from now on, the arm length of the polyp will be the "unity". This means that every length will be divided by s, the length of the polyps arms . Thus the arm of the polyp will have size one and the sizes of the paths will be rational numbers between 0 and 2. To make expressions shorter a path of size $\frac{u}{v} + \varepsilon$ will denote a path containing $\frac{su}{2v} + 1$ edges. Since there is no difference between two paths of the same size b_i or l_i, they denote both the path and the length of the path.

1.4 Worst Case of First Fit for Even Number of Arms

THEOREM 1.2 *If $p \geq 4$ is even then for every list L*

$$FF_p(L) \leq OPT_p(L)\frac{17}{10} + 4.$$

PROOF: The proof was inspired by the work of Johnson et al. [55]. They defined a weight function on the items. Their function is denoted now by $W'(x)$. We use a similar weight function $W(x)$ on the paths. In fact,

$$W(x) = \frac{2}{p}W'\left(\frac{x}{2}\right).$$

Explicitly,

$$W(x) = \begin{cases} \frac{2}{p}\frac{6}{10}x & \text{if } 0 < x \leq \frac{1}{3}, \\ \frac{2}{p}\frac{9}{10}x - \frac{2}{p}\frac{1}{10} & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3}, \\ \frac{2}{p}\frac{6}{10}x + \frac{2}{p}\frac{1}{10} & \text{if } \frac{2}{3} \leq x \leq 1, \\ \frac{2}{p} & \text{if } 1 < x \leq 2 \end{cases}$$

(see Fig 1.2). This means that

$$W\left(\frac{1}{3}\right) = \frac{2}{p}\frac{1}{2}, \quad W\left(\frac{2}{3}\right) = \frac{2}{p}\frac{1}{5}, \quad W(1) = \frac{2}{p}\frac{7}{10}.$$

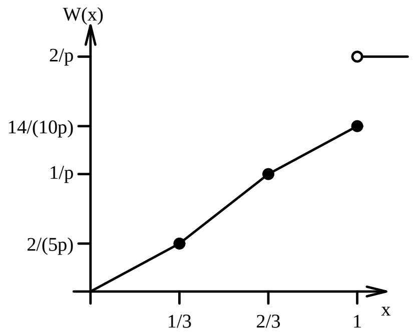

Fig. 1.2

First we are going to prove a few Lemmas. These are also similar to the ones in [55] but sometimes more complex.

LEMMA 1.3 *Let a polyp be filled with paths $l_1, l_2, \ldots, l_t$. Then*

$$\sum_{i=1}^{t} W(l_i) \leq \frac{17}{10}.$$

PROOF: If a polyp is filled with the above paths then some of the paths may cover some parts of two arms of the polyp. If this is the case then these two arms are called *paired*. We define a complete matching on the arms. If there are some paired arms they will be pairs in the matching too. The number of unpaired arms must be even since the number of arms is even. An arbitrary matching can be made between the leftover arms. In this way, a pair of arms filled with some paths is exactly the same as a bin filled with some items. But the size of the two arms is 2 here so if we want the bin to be size 1 then we must have items of size half of the corresponding path.

It is proved in [55] that if a bin of size 1 is filled with items $b_1, b_2, \dots, b_m$ then

$$\sum_{i=1}^{m} W'(b_i) \le \frac{17}{10}.$$

Let $h_1, \dots, h_m$ denote the lengths of paths put in a specific pair of arms. Here $b_i = \frac{h_i}{2}$, so in one pair of arms

$$\sum_{i=1}^{m} W(h_i) = \sum_{i=1}^{m} \frac{2}{p} W'(b_i) \le \frac{2}{p}\frac{17}{10}.$$

There are exactly $\frac{p}{2}$ pairs of arms so if we sum the weights in all pairs we obtain

$$\sum_{i=1}^{t} W(l_i) \le \frac{17}{10}. \qquad \blacksquare$$

Let us define the *coarseness* of a polyp. The coarseness of the polyp P_d is the largest α_d such that after completing the First Fit algorithm an additional path of size α_d would fit to some polyp with smaller index than d. The coarseness of P_1 is 0. Therefore if the First Fit algorithm puts a path of size l to polyp P_d then $l > \alpha_d$.

LEMMA 1.4 *Let a polyp P_d of coarseness $\alpha_d \le 1$ be filled with paths $l_1 \ge l_2 \ge \dots \ge l_t$ with the completed First Fit algorithm. If $\sum_{i=1}^{t} W(l_i) < 1$, then $\alpha_{d+1} > \alpha_d$.*

PROOF: Unfortunately it is not so easy to use the Lemmas of [55] to prove the present Lemma like in the previous case. Here we have to go through all the cases, again. Start with an observation which is used in each case.

There must be a pair of arms where the sum of the weights is less then $\frac{2}{p}$ otherwise the overall weight sum should be over 1 since there are $\frac{p}{2}$ pairs. Let A denote this pair of arms and $b_1, b_2, \dots, b_m$ the paths in it. If there is a path of size over 1 in A then the weight sum in this pair would be larger than $\frac{2}{p}$. So all paths in A are at most 1. We will show in each case that a path of size larger than α_d fits into this pair of arms.

Case 1: $\alpha_d \le \frac{1}{3}$.

For the proof it is sufficient to use the following lower estimate of $W(x)$ in $(0, 1]$, which may be seen on Fig. 1.3:

$$W(x) \ge \frac{2}{p}\frac{6}{10}x.$$

In this way

$$\sum_A b_i \frac{2}{p}\frac{6}{10} \le \sum_A W(b_i) < \frac{2}{p}$$

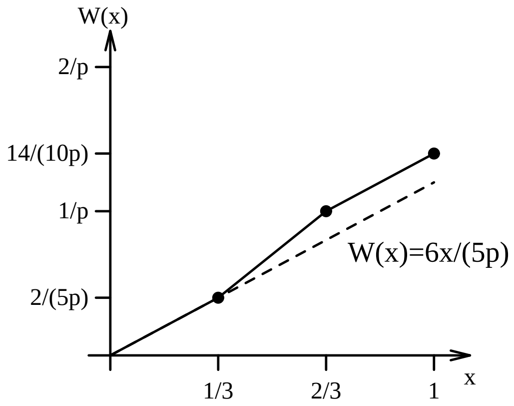

Fig. 1.3

is obtained which implies

$$\sum_A b_i < \frac{10}{6} = \frac{5}{3}.$$

This means that the empty space in this pair of arms is larger than $\frac{1}{3}$. If the paths are packed with the First Fit algorithm then a new path is always placed such that one of its endvertex is the endvertex of some earlier placed path except if it is placed into an empty arm. Anyway it is clear that there may not be two separated empty parts in a pair of arms. Thus the previous inequality shows that a path of size $> \frac{1}{3}$ will fit to this polyp. Since $\alpha_d \leq \frac{1}{3}$ this gives that $\alpha_{d+1} > \alpha_d$.

Case 2: $\frac{1}{3} < \alpha_d \leq \frac{2}{3}$.

By the definition of coarseness it is known that $b_i > \alpha_d > \frac{1}{3}$ holds for all b_i. Suppose there are q paths of size in $(\frac{1}{3}, \frac{2}{3}]$ and r paths of size in $(\frac{2}{3}, 1]$. The sets of these paths is denoted by Q and R, respectively. The following inequalities are clearly true:

$$\sum_A W(b_i) = \frac{2}{p}\frac{9}{10}\left(\sum_Q b_i\right) - q\frac{2}{p}\frac{1}{10} + \frac{2}{p}\frac{6}{10}\left(\sum_R b_i\right) + r\frac{2}{p}\frac{1}{10} < \frac{2}{p},$$

$$\frac{9}{10}\left(\sum_Q b_i\right) + \frac{6}{10}\left(\sum_R b_i\right) + \frac{r-q}{10} < 1,$$

$$\frac{3}{10}\left(\sum_Q b_i\right) + \frac{6}{10}\left(\sum_A b_i\right) + \frac{r-q}{10} < 1,$$

$$\sum_A b_i < \frac{10}{6} + \frac{q-r}{6} - \frac{1}{2}\sum_Q b_i.$$

So the empty space in A is more than

$$S = \frac{1}{3} + \frac{r-q}{6} + \frac{1}{2}\sum_Q b_i.$$

It will be proved that $S \geq \alpha_d$.

Our first observation is that $b_i > \alpha_d$ implies

$$S > \frac{1}{3} + \frac{r-q}{6} + \frac{1}{2}q\alpha_d.$$

In the case $q \geq 2$ one can verify that the trivial inequality

$$q \geq 2 \geq \frac{\alpha_d - \frac{1}{3} - \frac{r}{6}}{\frac{1}{2}\left(\alpha_d - \frac{1}{3}\right)}$$

implies $S > \alpha_d$.

It is easy to see that $0 \leq r \leq 1$ otherwise the weight sum in A would be at least $\frac{2}{p}$ since $W\left(\frac{2}{3}\right) = \frac{2}{p}\frac{1}{2}$. So there are only a few cases left.

If $q = 0$ and $r = 0$ or $r = 1$ then a paths of size 1 fits clearly into A. If $q = 1$ and $r = 0$ then, again, a path of size 1 fits into A. If $q = 1$ and $r = 1$ then

$$S \geq \frac{1}{3} + \frac{1}{2}\alpha_d \geq \alpha_d$$

since $\frac{1}{3} \geq \frac{1}{2}\alpha_d$.

Case 3: $\frac{2}{3} < \alpha_d$.

This is the easiest part. If there is at most one path in A it must be at most 1, therefore a path of size 1 fits into A. On the other hand if there are two paths then the weight sum is at least $\frac{2}{p}$ since both paths must be over $\frac{2}{3}$ by the definition of α_d and $W\left(\frac{2}{3}\right) = \frac{2}{p}\frac{1}{2}$. ■

LEMMA 1.5 *Let a polyp P_d of coarseness $\alpha_d \leq 1$ be filled with paths $b_1 \geq b_2 \geq \ldots \geq b_t$ in the completed First Fit algorithm. If $\sum_{i=1}^{t} W(b_i) = 1 - \beta_d \quad (0 < \beta_d < 1)$ then $\alpha_{d+1} \geq$ $\min(\alpha_d + \frac{5}{9}\beta_d, 1)$.*

PROOF: Suppose that $\alpha_{d+1} < \alpha_d + \frac{5}{9}\beta_d$. Then no path of length $\alpha_d + \frac{5}{9}\beta_d$ can fit to P_d when the list of paths L was packed to the polyps with the First Fit algorithm. A modified list L' will be created in the following way. Some of the paths in P_d will be enlarged.

It is possible that some of the paths cover vertices from two different arms. These two arms will be called paired, again. In a pair of arms the empty part is "together" because the First Fit algorithm was used to pack the paths. Let e denote the length of the empty part. This empty part does not contain the center, so it is contained in one arm completely. This arm contains a part of the path which covers the center and it must contain some paths which are as far from the center as possible. Let b_e be the closest path to the center among the latter paths. This path will be enlarged, let $b_e' = b_e + \min(\frac{5}{9}\beta_d, e)$. In this way the size of this empty part decreases by $\frac{5}{9}\beta_d$ or it becomes 0.

There may be some unpaired arms also. The empty part in these arms are attached to the center and all paths in the arms are as far from the center as possible. (There must be at least one path in each arm.) Let $e_1 \geq e_2 \geq \ldots \geq e_k$ be the sizes of the empty parts in the unpaired arms, b_i the closest path to the center in arm A_i. By our assumption $\alpha_d + \frac{5}{9}\beta_d > e_1$. If $e_1 < \alpha_d + \frac{5}{18}\beta_d$ then let $b_i' = b_i + \min(\frac{5}{18}\beta_d, e_i)$ for all i. If $e_1 \geq \alpha_d + \frac{5}{18}\beta_d$ then $\frac{5}{18}\beta_d > e_2 \geq e_3 \geq \ldots$ holds since $e_1 + e_2 < \alpha_d + \frac{5}{9}\beta_d$. Let $b_1' = b_1 + \min(\frac{5}{9}\beta_d - e_2, e_1)$ and $b_i' = b_i + e_i$ for $i = 2, 3, \ldots, k$.

The modified list $L' = b_1', b_2', \ldots$ is obtained by replacing the above paths by the enlarged paths. If L' is packed with the First Fit algorithm then every path will get the same place as the corresponding path in L, because no path fits to an earlier place (therefore α_d does not change), but every path fits to its "old" place. On the other hand it is easy to see that after packing L' there will be not enough empty space to insert an additional path of size $> \alpha_d$.

The slope of W in the interval $(0,1]$ is at most $\frac{2}{p}\frac{9}{10}$ and the sum of the enlargements is $\leq \frac{p}{2}\frac{5}{9}\beta_d$. Hence

$$\sum W(b_i') \leq \sum W(b_i) + \left(\frac{2}{p}\frac{9}{10}\right)\frac{p}{2}\frac{5}{9}\beta_d = 1 - \frac{\beta_d}{2} < 1$$

is obtained. Therefore L' would be a counter example to Lemma 1.4 which is a contradiction.

The proof is similar if $\alpha_d + \frac{5}{9}\beta_d > 1$. ■

Now we are prepared to complete the proof of Theorem 1.2.

Proof of Theorem 1.2: Let $Q_1, \ldots, Q_f$ denote the polyps for which $\sum_{i=1}^{t} W(b_j) = 1 - \beta_i$ holds with $\beta_i > 0$ in the First Fit algorithm in the same order as they appear in the list of polyps. The other polyps have weight sum at least 1. Let γ_i be the coarseness of Q_i. Suppose $1 \leq \gamma_{f-1} \leq \gamma_f$. Consequently all paths in Q_{f-1} are larger than 1, therefore all paths have weight $\frac{2}{p}$. Hence the number of paths in Q_{f-1} is $\leq \frac{p}{2} - 1$ since the weight sum is less than 1. This means that there must be a completely empty pair of arms. Therefore Q_{f-1} must be the last nonempty polyp, since any further path would fit to Q_{f-1}. This contradicts our assumption. Hence $\gamma_1 \leq \ldots \leq \gamma_{f-1} < 1$.

Lemma 1.5 and the definition of coarseness imply

$$\gamma_{i+1} \geq \gamma_i + \frac{5}{9}\beta_i \quad \text{for} \quad 1 \leq i \leq f-2.$$

As $\beta_{f-1}, \beta_f \leq 1$ thus

$$\sum_{i=1}^{f}\beta_i \leq \frac{9}{5}\sum_{i=1}^{f-2}(\gamma_{i+1} - \gamma_i) + 2 = \frac{9}{5}(\gamma_{f-1} - \gamma_1) + 2 < 4$$

holds.

Summing the weights for all paths $l_1, \ldots, l_m$ in L

$$FF_p(L) - 4 \leq \sum_{i=1}^{FF_p(L)-f} 1 + \sum_{i=1}^{f} 1 - \beta_i \leq \sum_{j=1}^{m} W(l_j)$$

is obtained. Finally, applying Lemma 1.3 we conclude

$$FF_p(L) \leq \sum_{i=1}^{m} W(l_i) + 4 \leq OPT_p(L)\frac{17}{10} + 4,$$

completing the proof. ■

Theorem 1.6 *If the arm size of the polyp is suitably large there exists a list L for any large enough M such that $OPT_p(L) \geq M$ and*

$$FF_p(L) \geq 1.6904 \cdot OPT_p(L).$$

PROOF: Let N be a positive integer divisible by 42 and let ε a be a suitably small rational number. The list of paths denoted by L consists of the following regions of paths in the order of appearance

a) $N\frac{p}{2}$ paths of size $\frac{1}{21} - 3\varepsilon$,
b) $N\frac{p}{2}$ paths of size $\frac{2}{7} + \varepsilon$,
c) $N\frac{p}{2}$ paths of size $\frac{2}{3} + \varepsilon$,
d) $N\frac{p}{2}$ paths of size $1 + \varepsilon$.

If these paths are packed with the First Fit algorithm then $\frac{p}{2}$ pairs of arms of the first $\frac{N}{42}$ polyps will contain 42 paths of region a). The coarseness of the next polyp will be $42 \cdot 3\varepsilon$ which is chosen small enough to exclude that one of the later coming paths could fit in. Namely $42 \cdot 3\varepsilon < \frac{2}{7} + \varepsilon$ must be true. In this way all the paths in region b) will go to later polyps. Exactly 3 will fit to the first arm, then 3 to the second etc. and after all arms are filled there is not enough place in the center for another one. This makes $3p$ in one polyp. Thus region b) will fill up $\frac{N}{6}$ polyps. The coarseness of the next polyp will be $\frac{2}{7} - 2\varepsilon$ so no later paths will fit into these polyps. In the same way, region c) will fill up $\frac{N}{2}$ polyps. Finally, only one path of size $1 + \varepsilon$ may fit into every pair of arms in each polyp. This occupies N more polyps. In this way

$$N\left[\frac{1}{42} + \frac{1}{6} + \frac{1}{2} + 1\right] \geq N \cdot 1.6904$$

polyps were needed to pack these paths.

The optimal packing of the list L packs one path of each region to one pair of arms. This packing is optimal since the sum of the sizes of the paths is equal to the sum of the sizes of the polyps. ■

It is easy to improve this construction with the paths $1+\varepsilon, 2/3+\varepsilon, 2/7+\varepsilon, 2/43+\varepsilon, 2/1807+\varepsilon, \ldots$ but the improvement is only about 0.001 in the ratio. On the other hand the following Conjecture can be made.

CONJECTURE 1.7 *If the arm size of the polyp is suitably large there exists a list L for any large enough M such that $OPT_p(L) \geq M$ and*

$$FF_p(L) \geq 1.7 \cdot OPT_p(L).$$

1.5 Worst Case of First Fit for Odd Number of Arms

The method is almost the same as in the case of even arms but the arm without a pair makes some differences here. The more arms the less difference. For example if $p = 3$ the difference is quite big. But for $p = 1001$ the results are very close to the even case.

THEOREM 1.8 *If $p \geq 7$ is odd then for every list L*

$$FF_p(L) \leq OPT_p(L)\left[1 + \frac{7p^2 + 6p - 1}{10p^2 - 2p}\right] + 4.$$

For some values of p this gives

$$\begin{aligned} R_7 &< 1.807 \\ R_9 &< 1.783 \\ R_{11} &< 1.768 \\ R_{15} &< 1.750 \\ \lim_{i\to\infty} R_i &\le 1.7 \end{aligned}$$

PROOF: The weight function used in this case is the following (see Fig. 1.4:

$$W(x) = \begin{cases} \frac{6}{5p-1}x & \text{if } 0 < x \le \frac{1}{3}, \\ \frac{9p-3}{p(5p-1)}x - \frac{p-1}{p(5p-1)} & \text{if } \frac{1}{3} \le x \le \frac{2}{3}, \\ \frac{6}{5p-1}x + \frac{p-1}{p(5p-1)} & \text{if } \frac{2}{3} \le x \le 1, \\ \frac{2}{p-1} & \text{if } 1 < x \le 2. \end{cases}$$

This implies

$$W\left(\frac{1}{3}\right) = \frac{2}{5p-1}, \quad W\left(\frac{2}{3}\right) = \frac{1}{p}, \quad W(1) = \frac{7p-1}{p(5p-1)}.$$

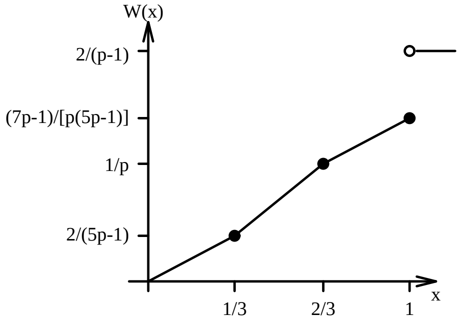

Fig. 1.4

The Lemmas needed to the proof are very similar to the even case.

LEMMA 1.9 *Let some polyp be filled with paths $l_1, l_2, \ldots, l_t$. Then*

$$\sum_{i=1}^{t} W(l_i) \le 1 + \frac{7p^2 + 6p - 1}{2p(5p-1)}.$$

PROOF: If a polyp is filled by the above paths then some of the paths may cover some parts of two arms of the polyp. If this is the case then these two arms are called *paired arms.* A maximal matching may be defined on the arms. If there are some paired arms they will be pairs in the matching, too. The number of unpaired arms must be odd since the number of arms is odd. An arbitrary matching is made between the leftover arms missing only one arm. This arm is called the *left arm.*

Let us first consider a pair of arms. If there is a path of size b such that $\frac{2}{3} < b \leq 1$ then replace it with two paths one with size $\frac{2}{3}$ and one with size $b - \frac{2}{3}$. The weight sum in the polyp was not changed by this modification because the slope of the weight function is the same in $(0, \frac{1}{3}]$ and in $[\frac{2}{3}, 1]$. Hence it may be supposed that there are no path of size in range $(\frac{2}{3}, 1]$ in the polyp. If there are two paths of size $b_1 \leq b_2 \leq \frac{1}{3}$ then they are replaced with a single path of size $b_1 + b_2$. In this way the weight sum of the polyp may increase but cannot decrease. Therefore it can be assumed that there are at most one path smaller than $\frac{1}{3}$ in every pair of arms. These replacements can be made in the left arm also, so the assumptions hold there too.

First it will be proved that the following inequality holds in a pair of arms A:

$$\sum_A W(b_i) \leq \frac{2}{p-1} + \frac{7p-1}{p(5p-1)} = U.$$

Let m be the number of paths in A. It is easy to see that $m \leq 6$.

$\mathbf{m} = 1$. The maximum weight of a path is $\frac{2}{p-1} < U$.

$\mathbf{m} = 2$. If $b_1 > 1$ then $b_2 \leq \frac{2}{3}$ thus

$$W(b_1) + W(b_2) \leq \frac{2}{p-1} + \frac{1}{p} < U.$$

If $\frac{2}{3} \geq b_1 \geq b_2$ then

$$W(b_1) + W(b_2) \leq 2\frac{1}{p} < U.$$

$\mathbf{m} = 3$. If $b_1 > 1$ and $\frac{2}{3} \geq b_2 \geq \frac{1}{3} \geq b_3$ then

$$\sum_A W(b_i) \leq \frac{2}{p-1} + W(\frac{2}{3}) + W(\frac{1}{3}) = U.$$

If $b_1 > 1$ and $\frac{2}{3} \geq b_2 \geq b_3 \geq \frac{1}{3}$ then $b_2 + b_3 < 1$ so

$$\sum_A W(b_i) = \frac{2}{p-1} + (b_2 + b_3)\frac{9p-3}{p(5p-1)} - 2\frac{p-1}{p(5p-1)} \leq \frac{2}{p-1} + \frac{7p-1}{p(5p-1)} = U.$$

If $\frac{2}{3} \geq b_1 \geq b_2 \geq b_3$ then

$$\sum_A W(b_i) \leq 3W\left(\frac{2}{3}\right) < U.$$

$\mathbf{m} = 4$. If $b_1 > 1$ then $\frac{2}{3} \geq b_2 \geq b_3 \geq \frac{1}{3}$ and $\frac{1}{3} > b_4$. Thus

$$\begin{aligned}\sum_A W(b_i) &= \frac{2}{p-1} + (b_2 + b_3)\frac{9p-3}{p(5p-1)} - 2\frac{p-1}{p(5p-1)} + b_4\frac{6}{5p-1} = \\ &= \frac{2}{p-1} + (b_2 + b_3 + b_4)\frac{9p-3}{p(5p-1)} - 2\frac{p-1}{p(5p-1)} - b_4\frac{3p-3}{p(5p-1)} < U,\end{aligned}$$

since $b_2 + b_3 + b_4 < 1$ and $b_4 \geq 0$.

If $\frac{2}{3} \geq b_1 \geq b_2 \geq b_3 \geq b_4 \geq \frac{1}{3}$ then

$$\begin{aligned}&\sum_A W(b_i) = (b_1 + b_2 + b_3 + b_4)\frac{9p-3}{p(5p-1)} - 4\frac{p-1}{p(5p-1)} \leq \\ &\leq 2\frac{9p-3}{p(5p-1)} - 4\frac{p-1}{p(5p-1)} < U,\end{aligned}$$

since this inequality is equivalent to the trivial $1 < 3p^2 + 6p$.
If $\frac{2}{3} \geq b_1 \geq b_2 \geq b_3 \geq \frac{1}{3} > b_4$ then

$$\sum_A W(b_i) = (b_1 + b_2 + b_3)\frac{9p-3}{p\,(5p-1)} - 3\frac{p-1}{p\,(5p-1)} + b_4\frac{6}{5p-1} =$$
$$= (b_1 + b_2 + b_3 + b_4)\frac{9p-3}{p\,(5p-1)} - 3\frac{p-1}{p\,(5p-1)} - b_4\frac{3p-3}{p(5p-1)} < U,$$

since $b_1 + b_2 + b_3 + b_4 \leq 2$, $b_4 \geq 0$ and $1 < p^4 + 4p$.

m $= 5$. If $b_1 > 1$ then three of the four other paths must be at least $\frac{1}{3}$ because there is only one path smaller than $\frac{1}{3}$. This is a contradiction. The rest of the cases can be proved with exactly the same argument as the corresponding cases in the previous paragraph.

m $= 6$. Only one case is possible: $\frac{2}{3} \geq b_1 \geq \ldots \geq b_5 \geq \frac{1}{3} \geq b_6$. This can be proved in the same way as above.

It remained to prove that in the left arm A_l

$$\sum_{A_l} W(b_i) \leq \frac{7p-1}{p(5p-1)} = U'$$

holds. Let $1-\varepsilon$ be the sum of the sizes in A_l. If $\varepsilon > 0$ then suppose that the inequality does not hold and that there is pair of arms A that contain the same paths as in A_l and an additional path b_0 of length > 1. Then

$$\sum_A W(b_i) > \frac{2}{p-1} + \frac{7p-1}{p(5p-1)} = U,$$

a contradiction to the previous case. Therefore the inequality is true for any $\varepsilon > 0$. The weight function is continuous in $(0, 1]$ therefore it must be also true for $\varepsilon = 0$.

To finish the proof of the Lemma notice that there are $\frac{p-1}{2}$ pairs of arms and one left arm, thus

$$\sum_{i=1}^{t} W(l_i) \leq \frac{p-1}{2}U + U' =$$
$$= \frac{p-1}{2}\left[\frac{2}{p-1} + \frac{7p-1}{p(5p-1)}\right] + \frac{7p-1}{p(5p-1)} = 1 + \frac{7p^2+6p-1}{2p(5p-1)}. \qquad \blacksquare$$

Lemma 1.10 *Let a polyp P_d of coarseness $\alpha_d \leq 1$ be filled with paths $l_1 \geq l_2 \geq \ldots \geq l_t$ with the completed First Fit algorithm. If $p \geq 7$ and $\sum_{i=1}^{t} W(l_i) < 1$, then $\alpha_{d+1} > \alpha_d$.*

Proof: Let A be the pair of arms where the weight sum is minimal among the pairs. Suppose that no path of size $> \alpha_d$ fits to the polyp P_d after completing the First Fit algorithm, therefore such a path does not fit neither to A nor to the left arm A_l. It will be proved in each case that this implies

$$\sum_{i=1}^{t} W(l_i) \geq 1,$$

a contradiction.

If there is a path of size >1 in A then

$$\sum_{i=1}^{t} W(l_i) \geq \frac{p-1}{2}\frac{2}{p-1} = 1$$

holds. Since a path of size >1 cannot fit to A_l it may be assumed that there is no path >1 in the polyp.

Case 1: $\alpha_d \leq \frac{1}{3}$.

It is easy to see on Fig. 1.5 that $W(x) \geq \frac{6}{5p-1}x$ holds in $(0,1]$

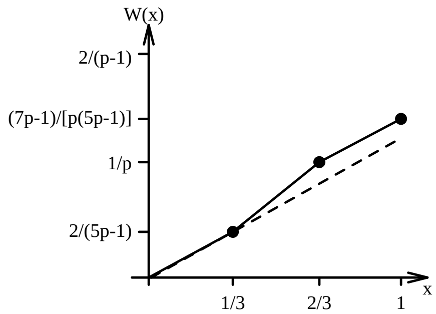

Fig. 1.5

Therefore

$$\sum_{A} W(b_i) > (2-\alpha_d)\frac{6}{5p-1} \quad \text{and} \quad \sum_{A_l} W(b_i) > (1-\alpha_d)\frac{6}{5p-1}$$

holds, since the sum of the lengths must be $>(2-\alpha_d)$ in A and $>(1-\alpha_d)$ in A_l.

All the other pairs have weight sums at least as much as A, so

$$\sum_{i=1}^{t} W(l_i) > \frac{p-1}{2}(2-\alpha_d)\frac{6}{5p-1} + (1-\alpha_d)\frac{6}{5p-1}$$

holds. It is easy to see that this is at least 1 if $\alpha_d \leq \frac{1}{3}$, a contradiction.

Case 2: $\frac{1}{3} < \alpha_d \leq \frac{2}{3}$.

Let m be the number of paths in A and k the number of paths in A_l. Clearly $1 \leq m \leq 5$ and $1 \leq k \leq 2$. Let $b_1 \geq \ldots \geq b_m$ denote the paths in A and $a_1 \geq \ldots \geq a_k$ $(k \geq 1)$ the paths in A_l.

$\mathbf{m} = 1$. In this case $b_1 > 1$, since otherwise a path of size $>\alpha_d$ would fit into A, and this case is proved above.

$\mathbf{m} = 2$. If $b_1 \geq b_2 \geq \frac{2}{3}$ then

$$\sum_{i=1}^{t} W(l_i) \geq \left[\frac{6}{5p-1}(b_1+b_2) + 2\frac{p-1}{p(5p-1)}\right]\frac{p-1}{2} + W(a_1) >$$

$$> \left[\frac{6}{5p-1}(2-\alpha_d) + 2\frac{p-1}{p(5p-1)}\right]\frac{p-1}{2} + \frac{9p-3}{p\,(5p-1)}\alpha_d - \frac{p-1}{p\,(5p-1)} =$$

$$= \frac{-3p^2+12p-3}{p\,(5p-1)}\alpha_d + \frac{7p^2-9p+2}{p\,(5p-1)} \geq 1,$$

since $b_1 + b_2 > 2 - \alpha_d$, $a_1 > \alpha_d$, the coefficient of α_d is negative if $p \geq 3$ and $\alpha_d \leq \frac{2}{3}$.
If $b_1 \geq \frac{2}{3} \geq b_2 > \alpha_d > \frac{1}{3}$ then

$$\sum_{i=1}^{t} W(l_i) \geq \left[\frac{6}{5p-1}b_1 + \frac{9p-3}{p(5p-1)}b_2\right]\frac{p-1}{2} + W(\alpha_d) >$$
$$> \left[\frac{6}{5p-1}(2-\alpha_d) + \frac{3p-3}{p(5p-1)}\alpha_d\right]\frac{p-1}{2} + W(\alpha_d) \geq 1$$

using the same observations as before.
If $\frac{2}{3} > b_1 \geq b_2$ then a path of size $> \alpha_d$ fits to A which is a contradiction.

m = 3. If $b_1 = b_2 = b_3$ then

$$\sum_{i=1}^{t} W(l_i) \geq \frac{3}{p}\frac{p-1}{2} \geq 1$$

if $p \geq 3$.
The case $b_1 \geq b_2 \geq \frac{2}{3} > b_3 > \alpha_d$ can be proved exactly like the case $m = 2$, $b_1 \geq \frac{2}{3} \geq b_2 > \alpha_d$.
If $b_1 \geq \frac{2}{3} > b_2 \geq b_3 > \alpha_d$ then

$$\sum_{i=1}^{t} W(l_i) \geq$$
$$\geq \left[\frac{6}{5p-1}b_1 + \frac{9p-3}{p(5p-1)}(b_2+b_3) - \frac{p-1}{p(5p-1)}\right]\frac{p-1}{2} + W(\alpha_d) =$$
$$= \left[\frac{6}{5p-1}(b_1+b_2+b_3) + \frac{3p-3}{p(5p-1)}(b_2+b_3) - \frac{p-1}{p(5p-1)}\right]\frac{p-1}{2} + W(\alpha_d) >$$
$$> \left[\frac{6}{5p-1}(2-\alpha_d) + \frac{3p-3}{p(5p-1)}(2\alpha_d) - \frac{p-1}{p(5p-1)}\right]\frac{p-1}{2} + \frac{6}{5p-1}(1-\alpha_d)$$

holds, since $\sum_{i=1}^{k} a_i > 1 - \alpha_d$ and $W(x) \geq \frac{6}{5p-1}x$ holds in $(0,1]$ (see Fig. 1.5). The coefficient of α_d is positive in this expression, so it takes its minimum if $\alpha_d = \frac{1}{3}$. It is easy to check that the expression is ≥ 1 in this case if $p \geq 3$.
If $\frac{2}{3} > b_1 \geq b_2 \geq b_3 > \alpha_d \geq \frac{1}{2}$ then

$$\sum_{i=1}^{t} W(l_i) \geq \left[\frac{9p-3}{p(5p-1)}(b_1+b_2+b_3) - 3\frac{p-1}{p\,(5p-1)}\right]\frac{p-1}{2} + W\,(\alpha_d) \geq$$
$$\geq \left[\frac{9p-3}{p(5p-1)}(3\alpha_d) - 3\frac{p-1}{p\,(5p-1)}\right]\frac{p-1}{2} + \frac{9p-3}{p\,(5p-1)}\alpha_d - \frac{p-1}{p\,(5p-1)} \geq 1$$

holds if $p \geq 7$, since the coefficient of α_d is positive and $\alpha_d \geq \frac{1}{2}$.
If $\frac{2}{3} > b_1 \geq b_2 \geq b_3 > \alpha_d$ and $\alpha_d < \frac{1}{2}$ then we first prove that the sum of the weights in A_l is $\geq \frac{7p-1}{2p(5p-1)}$. The sum of the sizes is $> 1 - \alpha_d$ hence there is a path of size $\geq \frac{1}{2}$ or there are two paths both $> \frac{1}{3}$. It is easy to check that our claim is true in both cases. Therefore we obtain

$$\sum_{i=1}^{t} W(l_i) \geq \left[\frac{9p-3}{p(5p-1)}(b_1+b_2+b_3) - 3\frac{p-1}{p\,(5p-1)}\right]\frac{p-1}{2} + \frac{7p-1}{2p\,(5p-1)} \geq$$
$$\geq \left[\frac{9p-3}{p(5p-1)}(2-\alpha_d) - 3\frac{p-1}{p\,(5p-1)}\right]\frac{p-1}{2} + \frac{7p-1}{2p\,(5p-1)} \geq 1,$$

if $p \geq 7$ since $b_1 + b_2 + b_3 > 2 - \alpha_d$, the coefficient of α_d is negative and $\alpha_d < \frac{1}{2}$. (Note that these are the only cases which are not true for $p = 3, 5$.)

m $= 4$. The case $b_1 \geq b_2 \geq \frac{2}{3} > b_3 \geq b_4 > \alpha_d$ can be proved exactly like the case $m = 2$, $b_1 \geq \frac{2}{3} > b_2 > \alpha_d$ and the case $b_1 \geq \frac{2}{3} > b_2 \geq b_3 \geq b_4 > \alpha_d$ exactly like the case $m = 3$, $b_1 \geq \frac{2}{3} > b_2 \geq b_3 > \alpha_d$.

If $\frac{2}{3} > b_1 \geq b_2 \geq b_3 \geq b_4 > \alpha_d$ then the proof is similar to the corresponding proof in the previous case. In this case

$$\sum_{i=1}^{t} W(l_i) \geq \left[\frac{9p-3}{p(5p-1)}(b_1 + b_2 + b_3 + b_4)\right]\frac{p-1}{2} + W\left(\frac{1}{3}\right)$$

holds. This expression takes its minimum if $b_1 + b_2 + b_3 + b_4$ achieves its minimum. $b_1 + b_2 + b_3 + b_4 > 4\alpha_d$ and $b_1 + b_2 + b_3 + b_4 > 2 - \alpha_d$ therefore the minimum, $\frac{8}{5}$, is taken when $\alpha_d = \frac{2}{5}$. Then it easy to see that the above expression is > 1.

m $= 5$. The only possible case is $\frac{2}{3} > b_1 \geq \ldots \geq b_5 > \alpha_d$. Then the lower bound $W(x) \geq \frac{6}{5p-1}$ is used again (see Fig. 1.5). All the paths in the paired arms have length $> \alpha_d$ and the sum of the sizes in A_l is $> (1 - \alpha_d)$, therefore

$$\sum_{i=1}^{t} W(l_i) > \frac{p-1}{2}\frac{6}{5p-1}5\alpha_d + \frac{6}{5p-1}(1 - \alpha_d) = \alpha_d\frac{15p-21}{5p-1} + \frac{6}{5p-1} \geq 1$$

is obtained since $\alpha_d > \frac{1}{3}$.

Case 3: $\frac{2}{3} < \alpha_d \leq 1$.

By the definition of the coarseness every path in the polyp must be larger than $\frac{2}{3}$. The sum of the sizes in a pair of arms must be $> \frac{4}{3}$, therefore there must be at least two paths in each pair. Similarly there must be at least one paths in A_l. Thus $\sum_{i=1}^{t} W(l_i) > pW\left(\frac{2}{3}\right) = 1$ is obtained completing the proof of the Lemma. ■

LEMMA 1.11 *Let a polyp P_d of coarseness $\alpha_d \leq 1$ be filled with paths $l_1 \geq l_2 \geq \ldots \geq l_t$ in the completed First Fit algorithm. If $\sum_{i=1}^{t} W(l_i) = 1 - \beta \quad (0 < \beta < 1)$ then $\alpha_{d+1} \geq$* $\min(\alpha_d + \frac{5}{9}\beta, 1)$.

The proof of this Lemma is very similar to the proof of Lemma 1.5 so it is left to the reader. ■

The rest of the proof of Theorem 1.8 is almost the same as the proof of Theorem 1.2 so it is also left to the reader. ■

By slight modification in the weight function we can prove that $R_3 < 2$ and $R_5 < 1.858$ which does not differ very much from the value of the formula in Theorem 1.8 for $p = 3$ and $p = 5$.

THEOREM 1.12 *If the arm size of the polyp is suitably large then there exists a list L for any large enough M such that $OPT_p(L) \geq M$ and*

$$FF_p(L) \geq OPT_p(L)\left[\frac{p+1}{42p} + \frac{p+1}{6p} + \frac{p+1}{2p} + 1\right].$$

For some values of p this gives the following lower bounds for the ratio (the upper bound is from Theorem 1.8).

$$\begin{aligned} 1.920 &< R_3(< 2) \\ 1.828 &< R_5(< 1.858) \\ 1.789 &< R_7(< 1.807) \\ 1.736 &< R_{15}(< 1.750) \\ 1.6904 &< \lim_{i\to\infty} R_i(\le 1.7) \end{aligned}$$

The lower and upper bounds are quite close.

PROOF: This proof is also very similar to the proof of Theorem 1.6 but it is worth making the differences clear.

Let N be a positive integer divisible by $42p$ and let ε a be a suitably small rational number. The list of paths denoted by L consists of the following regions of paths in the order of appearance

a) $N\frac{p-1}{2}$ paths of size $\frac{1}{21} - 3\varepsilon$,
a') N paths of size $\frac{1}{21} - 2\varepsilon$,
b) $N\frac{p+1}{2}$ paths of size $\frac{2}{7} + \varepsilon$,
c) $N\frac{p+1}{2}$ paths of size $\frac{2}{3} + \varepsilon$,
d) $N\frac{p-1}{2}$ paths of size $1 + \varepsilon$.

If these paths are packed with the First Fit algorithm then 42 paths from a) and a') fits to a pair of arms and 21 to the left arm. So

$$\frac{N\frac{p-1}{2} + N}{42\frac{p-1}{2} + 21} = N\frac{p+1}{42p}$$

polyps are needed to pack a) and a'). Similarly one can see that

$$N\frac{p+1}{6p} + N\frac{p+1}{2p}$$

additional polyps are needed to pack paths from b) and c). Note that as in the even case the coarseness of the polyps are increasing. Finally the main difference comes from the fact that only $\frac{p-1}{2}$ paths of size $1+\varepsilon$ fits to one polyp. So we need N more polyps to finish the packing.

On the other hand it is clear that these paths fit to N polyps and this is an optimal packing. ■

There is one more construction worth mentioning. This is an other lower bound which gives a somewhat better result for $p = 3$ than Theorem 1.12, but for larger number of arms Theorem 1.12 is better.

THEOREM 1.13 *If the arm size of the polyp is suitably large then there exists a list L for any large enough M such that $OPT_p(L) \ge M$ and*

$$FF_p(L) \ge OPT_p(L)\left[\frac{p+1}{24p} + \frac{p+1}{7p-1} + \frac{p+1}{2p} + 1\right].$$

This gives $1.922 < R_3(< 2)$.

PROOF: Let N be a positive integer divisible by $24p$ and $7p-1$. The list L of paths consists of the following regions of paths in the order of appearance

a) $N\frac{p-1}{2}$ paths of size $\frac{1}{12}-3\varepsilon$

a') N paths of size $\frac{1}{12}-2\varepsilon$

b) $N\frac{p+1}{2}$ paths of size $\frac{1}{4}+\varepsilon$

c) $N\frac{p+1}{2}$ paths of size $\frac{2}{3}+\varepsilon$

d) $N\frac{p-1}{2}$ paths of size $1+\varepsilon$

The proof otherwise is the same as above so it is left to the reader. ∎

Chapter 2

Vertex Disjoint Polyp Packing

The reader can find the introduction and the notations for this Chapter in Section 1.1.

2.1 Worst Case of Vertex Disjoint First Fit

Let us define the algorithm which will be investigated. It is almost the same as in the edge disjoint case. The only difference is that we can use the center only once thus the algorithm is simpler.

Let the p–polyps be indexed as $P_1, P_2, \ldots$, initially each empty. The arms of P_i are also indexed $A_{i,1}, \ldots, A_{i,p}$. The paths $l_1, l_2, \ldots, l_t$ will be placed in that order. To place l_i, we try to put it into $A_{1,1}$ so that it does not contain the center. If it is not possible we try to put it into $A_{1,2}$ and so on. If we find an arm which is not "full" we put l_i to this arm as far as possible from the center. If all the arms of P_1 are "full" we try to put l_i to $A_{1,1}$ and $A_{1,2}$ so that l_i will contain the center of P_1. Then we try to put it into $A_{1,1}$ and $A_{1,3}$ and so on, we continue using the lexicographic order of the pair of arms. If we cannot put l_i to P_1, we move on to P_2, etc. This algorithm is the First Fit algorithm for the vertex disjoint case.

L, $FF_p(L)$, $OPT_p(L)$ and R_p denote the same notions as in Chapter 1.

To make computations easier from now on the arm length of the polyp will be the "unity" but this is defined slightly different from the edge disjoint case. To be precise every length will be divided by $s+1$, where s denotes the number of edges in an arbitrary arm of the polyp (all arms have the same size). In addition in this vertex disjoint case the length of a path, despite the usual notation, always denotes the number of vertices of the path. Suppose for example that a path contains e edges where $e \leq 2s$. The length of this path will be $\frac{e+1}{s+1}$ with the new notation. To make expressions shorter $\frac{u}{v} + \varepsilon$ will denote a path containing $(s+1)\frac{u}{v} + 1$ vertices and therefore $(s+1)\frac{u}{v}$ edges. Thus an arm of the polyp with the center as one endvertex will have size 1 and the sizes of the paths will be rational numbers between 0 and $2 - \varepsilon$. This notation assures that if there are some path such that the sum of their sizes is d then $d(s+1)$ consecutive vertices of an arm are needed to pack them to this arm. Since there is no difference between two path of the same size b_i or l_i, they denote both the path and the size of the path.

THEOREM 2.1 *For every list L and $p \geq 4$*

$$FF_p(L) \leq OPT_p(L)\left[\frac{p-1}{5p+1} + \frac{p-1}{2p+1} + \frac{p-1}{p} + 1\right] + 5.$$

For some values of p this gives

$$\begin{aligned} R_4 &< 2.227 \\ R_5 &< 2.318 \\ R_6 &< 2.380 \\ R_{15} &< 2.570 \\ \lim_{i\to\infty} R_i &\le 2.7 \end{aligned}$$

PROOF: A weight function, similar to the one used in Chapter 1, is defined on the paths, but this function is much more complicated.

$$W(x) = \begin{cases} \frac{6}{5p+1}x & \text{if } 0 < x \le \frac{1}{6}, \\ \frac{18p}{(5p+1)(2p+1)}x - \frac{p-1}{(5p+1)(2p+1)} & \text{if } \frac{1}{6} \le x \le \frac{1}{3}, \\ \frac{6}{5p+1}x + \frac{p-1}{(5p+1)(2p+1)} & \text{if } \frac{1}{3} \le x < \frac{1}{2}, \\ \frac{1}{p} & \text{if } \frac{1}{2} \le x \le \frac{2}{3}, \\ \frac{21p+6}{(2p+1)(5p+1)}x - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} & \text{if } \frac{2}{3} \le x < 1, \\ 1 & \text{if } 1 \le x \le 2. \end{cases}$$

To make the picture more clear the key points of the function are (see Fig. 2.1)

$$W\left(\tfrac{1}{6}\right) = \tfrac{1}{5p+1}, \qquad W\left(\tfrac{1}{3}\right) = \tfrac{1}{2p+1}, \qquad \lim_{\varepsilon\to 0} W\left(\frac{1}{2} - \varepsilon\right) = \frac{1}{5p+1} + \frac{1}{2p+1},$$

$$W(x) = \tfrac{1}{p} \text{ for } \tfrac{1}{2} \le x \le \tfrac{2}{3}, \qquad W\left(\tfrac{2}{3}\right) = \tfrac{1}{p}, \qquad \lim_{\varepsilon\to 0} W(1-\varepsilon) = \frac{1}{5p+1} + \frac{1}{2p+1} + \frac{1}{p}.$$

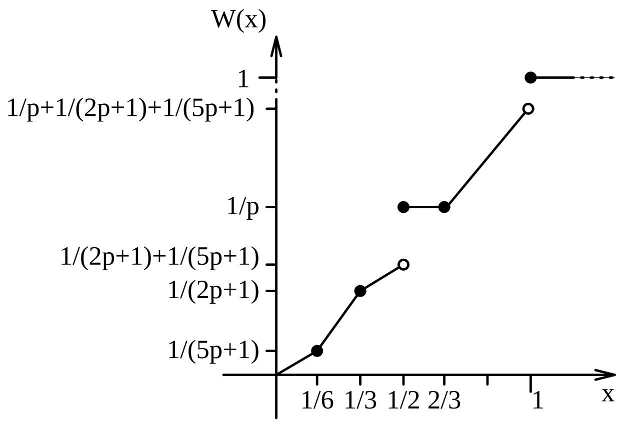

Fig. 2.1

LEMMA 2.2 *Let a polyp be filled with paths $l_1, l_2, \ldots, l_m$ in an arbitrary way. Then*

$$\sum_{i=1}^{m} W(l_i) \le \frac{p-1}{5p+1} + \frac{p-1}{2p+1} + \frac{p-1}{p} + 1.$$

PROOF: There will be several cases. The first case is when there is no path ≥ 1. Choose a path which covers vertices in two different arms. It may be supposed that it is l_1. This path < 1 so it cannot cover $\geq \frac{1}{2}$ in both arms. Suppose it covers $< \frac{1}{2}$ in the arm A_1 and some portion of A_2. Replace l_1 and all the other paths in A_2 by a path which has the same size as the sum of the replaced paths. Thus the length of the new path is $< \frac{3}{2}$ because ≤ 1 is contained by A_2 and $< \frac{1}{2}$ by A_1. If this size is < 1 replace it with a path of size 1. If there is no path which covers the center, replace the contents of any arm with a path of size 1.

One can check that if $0 < x < 1$ then $W(x)/x < \frac{2}{p}$ (see Fig. 2.2).

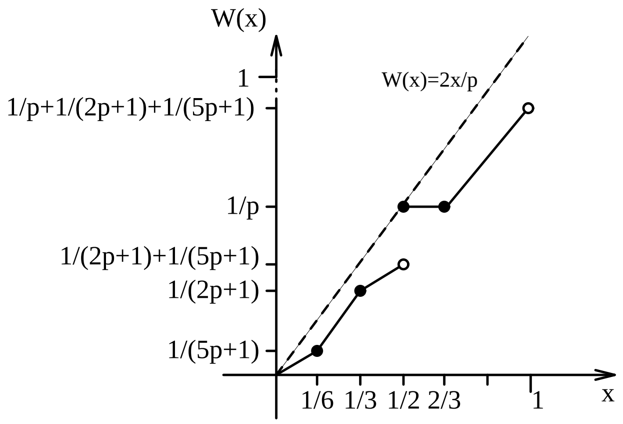

Fig. 2.2

The sum of the weights of the replaced paths was $< \frac{3}{2}\frac{2}{p}$. This is at most 1 if $p \geq 4$. Since the weight of the new path is 1, the overall weight sum in the polyp is increased. Therefore the proof is reduced to the second case when there is a path of size ≥ 1.

In the second case when there is a path of size ≥ 1, this path must contain the center. The other paths are entirely contained by the arms. Now we prove that if there are paths with sizes $b_1 \leq b_2, \leq \ldots \leq b_t$ such that $\sum_{i=1}^{t} b_i < 1$ then

$$\sum_{i=1}^{t} W(b_i) \leq \frac{1}{5p+1} + \frac{1}{2p+1} + \frac{1}{p}.$$

The first subcase is when $1 > b_1 \geq \frac{2}{3}$. In this case $b_i < \frac{1}{2}$ holds for all other paths. The slope of $W(x)$ in the interval $[\frac{2}{3}, 1)$ is larger than anywhere in $[0, \frac{1}{2})$. Replace all paths with a single path of size $\sum_{i=1}^{t} b_i$, then the weight sum increases. Since $W(x) < \frac{1}{5p+1} + \frac{1}{2p+1} + \frac{1}{p}$ if $x < 1$, this case is proved.

Suppose that there is no $b_i > \frac{2}{3}$ but $b_1 \geq \frac{1}{2}$. The slope of $W(x)$ in $[\frac{1}{2}, \frac{2}{3}]$ is less than anywhere in $[0, \frac{1}{2})$ thus the weight sum increases if b_1 is replaced by two paths of length $\frac{1}{2}$ and $b_1 - \frac{1}{2} < \frac{1}{2}$. Now it is sufficient to prove that if $\sum_{i=1}^{t} b_i \leq \frac{1}{2}$ then

$$\sum_{i=1}^{t} W(b_i) \leq \frac{1}{5p+1} + \frac{1}{2p+1},$$

since $W\left(\frac{1}{2}\right) = \frac{1}{p}$. W is almost the same in the interval $[0, \frac{1}{2})$ as the weight function in [55] and this part was basically proved there therefore the computations are omitted.

Only one case remains when $b_i < \frac{1}{2}$. However, $x < \frac{1}{2}$ implies

$$W(x)/x \leq \frac{3}{2p+1} < \frac{1}{5p+1} + \frac{1}{2p+1} + \frac{1}{p}.$$

To finish the proof note that if there is one path of length ≥ 1 (with weight 1) and $p-1$ arms, each containing paths with weight sum less than $\frac{1}{5p+1} + \frac{1}{2p+1} + \frac{1}{p}$, then

$$\sum_{i=1}^{m} W(l_i) \leq \frac{p-1}{5p+1} + \frac{p-1}{2p+1} + \frac{p-1}{p} + 1. \qquad \blacksquare$$

The coarseness of the polyp P_d is defined the same as in Chapter 1. It is the largest α_d such that after completing the First Fit algorithm an additional path of size α_d would fit to some polyp with smaller index than d. The coarseness of P_1 is 0, again. Note that if the First Fit algorithm puts a path of size l to polyp P_d then $l > \alpha_d$ holds like in the edge disjoint case.

LEMMA 2.3 *Let a polyp P_d of coarseness $\alpha_d < 1$ be filled with paths $b_1 \geq b_2 \geq \ldots \geq b_t$ in the completed First Fit algorithm. If $\sum_{i=1}^{t} W(b_i) < 1$ then $\alpha_{d+1} > \alpha_d$.*

PROOF: It may be supposed that there is no path ≥ 1, otherwise the sum of the weights would be at least 1.

There will be a number of cases depending on the value of α_d.

Case 1: $\alpha_d \leq \frac{1}{6}$.

On Fig. 2.3 the reader may see that $W(x) \geq \frac{6}{5p+1}x$ holds in $[0, 1)$ thus

$$1 > \sum_{i=1}^{t} W(b_i) \geq \frac{6}{5p+1} \sum_{i=1}^{t} b_i \quad \text{therefore}$$
$$\frac{5p+1}{6} > \sum_{i=1}^{t} b_i$$

is obtained.

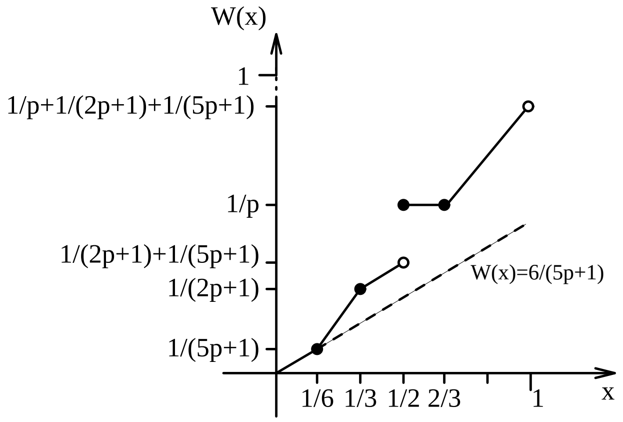

Fig. 2.3

If there is a path containing the center then one of the arms must be covered completely and the empty part is "together" in the other arms since every path is placed to the arm as far from the center as possible. Hence there are at most $p-1$ empty parts. The sum of the sizes of the empty parts is

$$> p - \frac{5p+1}{6} = \frac{p-1}{6},$$

thus there is an empty part with size $> \frac{p-1}{6}\frac{1}{p-1} = \frac{1}{6} \geq \alpha_d$. By definition this implies $\alpha_{d+1} > \alpha_d$.

If there is no path containing the center then we can "combine" the empty parts of two arms together — the empty parts must be "adjacent" to the center — so there are $p-1$ empty parts again, so the conclusion is the same.

Case 2: $\frac{1}{6} < \alpha_d \leq \frac{1}{4}$.

The definition of the coarseness implies $b_1, b_2, \ldots \geq \alpha_d > \frac{1}{6}$. It is clear from Fig. 2.4 that the line connecting $(0,0)$ and $(\alpha_d, W(\alpha_d))$ on the diagram of the weight function lies under the weight function in the interval $[\alpha_d, 1)$. The "worst case" is when $\alpha_d = \frac{1}{4}$ then the line contains $(\frac{1}{2}, W(\frac{1}{2}))$. Using this observation

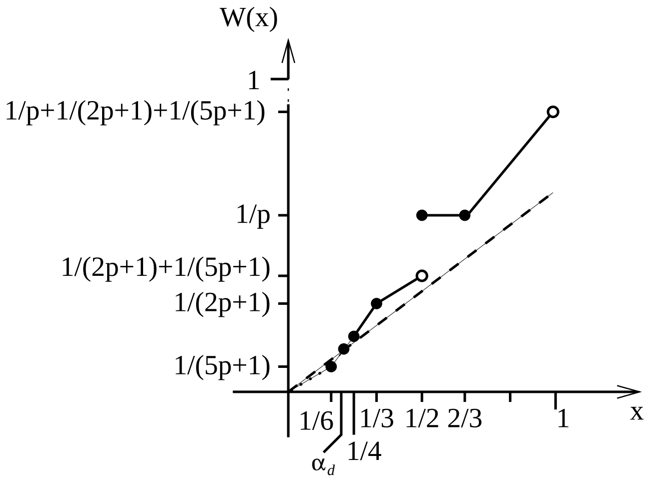

Fig. 2.4

$$1 > \sum_{i=1}^{t} W(b_i) \geq \frac{W(\alpha_d)}{\alpha_d} \sum_{i=1}^{t} b_i,$$

thus

$$\frac{\alpha_d}{W(\alpha_d)} > \sum_{i=1}^{t} b_i$$

holds. There are $p-1$ empty parts again, like in the previous case, so it remains to show that

$$\frac{p - \frac{\alpha_d}{W(\alpha_d)}}{p-1} > \alpha_d,$$

or equivalently

$$\frac{p(-18\alpha_d^2 + 9\alpha_d - 1)}{18\alpha_d p - p + 1} > 0,$$

which is clearly true if $\frac{1}{6} < \alpha_d \leq \frac{1}{4}$ and $p \geq 4$.

Case 3: $\frac{1}{4} < \alpha_d \leq \frac{1}{3}$, the center is not covered.

Suppose that $\alpha_{d+1} \not> \alpha_d$. It will be shown that this implies $\sum_{i=1}^{t} W(b_i) \geq 1$, a contradiction. Two major cases are distinguished. The central vertex may be covered by a path or it may remain uncovered. When it is covered there is a path which has some parts in two arms. These two arms will be called *paired*. If the central vertex is the endvertex of the covering path then any other arm may be paired with the arm containing the covering path.

If the central vertex is not covered then let s_1 and s_2 be the sum of the sizes of the covered parts in the two least filled arms. Since a path of size over α_d should not fit in, we obtain $s_1 + s_2 \geq 2 - \alpha_d$. On the other hand

$$s_1 + s_2 \leq \frac{2}{p} \sum_{i=1}^{t} b_i,$$

assuring that

$$p - p\frac{\alpha_d}{2} \leq \sum_{i=1}^{t} b_i.$$

It is clear from Fig. 2.5 that

$$W(x) \geq \left(\frac{2}{5p+1} + \frac{2}{2p+1} \right) x,$$

holds in $(\frac{1}{4}, 1)$.

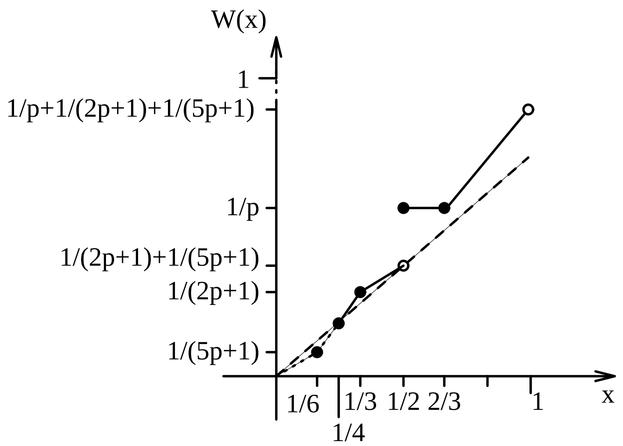

Fig. 2.5

Thus

$$\sum_{i=1}^{t} W(b_i) \geq \sum_{i=1}^{t} b_i \left(\frac{2}{5p+1} + \frac{2}{2p+1} \right) \geq \left(p - p\frac{\alpha_d}{2} \right) \left(\frac{2}{5p+1} + \frac{2}{2p+1} \right) > 1$$

holds, which is a contradiction.

The case when the center is covered is somewhat more complicated. Two subcases, $\frac{1}{4} < \alpha_d \leq \frac{2}{7}$ and $\frac{2}{7} < \alpha_d \leq \frac{1}{3}$ are distinguished.

Case 4: $\frac{1}{4} < \alpha_d \leq \frac{2}{7}$, the center is covered.

Suppose that $\alpha_{d+1} \not> \alpha_d$. It will be shown that this implies $\sum_{i=1}^{t} W(b_i) \geq 1$, a contradiction. In this case it will be shown that the sum of the weights in the paired arms together is at least $\frac{7}{3p+1}$ and the sum of the weights in the other arms is at least $\frac{3}{3p+1}$. The latter claim is proved first. If there is only one path in the arm then it must be larger than $\frac{2}{3}$ otherwise $\alpha_{d+1} > \alpha_d$. In this way its weight is at least $\frac{1}{p} > \frac{3}{3p+1}$.

Now a few subcases will be considered when there are two paths in the arm.

If one of the paths is $\geq \frac{1}{2}$ then the sum of the weights is again at least $\frac{1}{p} > \frac{3}{3p+1}$.

When both paths are $< \frac{1}{2}$ and $> \frac{1}{3}$ then $b_1 + b_2 \geq 1 - \alpha_d \geq 1 - \frac{2}{7}$ must hold, thus

$$W(b_1) + W(b_2) \geq \left(1 - \frac{2}{7}\right)\frac{6}{5p+1} + 2\frac{p-1}{(2p+1)(5p+1)} > \frac{3}{3p+1}.$$

If $\frac{1}{2} > b_1 > \frac{1}{3} \geq b_2 > \frac{2}{7} \geq \alpha_d > \frac{1}{4}$ then $b_1 + b_2 \geq 1 - \alpha_d$ must be true, therefore

$$\begin{aligned}
&W(b_1) + W(b_2) = \\
&= \frac{6}{5p+1}b_1 + \frac{p-1}{(2p+1)(5p+1)} + \frac{18p}{(2p+1)(5p+1)}b_2 - \frac{p-1}{(2p+1)(5p+1)} = \\
&= (b_1 + b_2)\frac{6}{5p+1} + b_2\frac{18p - 12p - 6}{(2p+1)(5p+1)} \geq \\
&\geq (1 - \alpha_d)\frac{6}{5p+1} + b_2\frac{6p-6}{(2p+1)(5p+1)} > \frac{3}{3p+1},
\end{aligned}$$

since $b_2 > \frac{2}{7}$, $\alpha_d \leq \frac{2}{7}$ and $p \geq 4$.

If $\frac{1}{2} > b_1 > \frac{1}{3} > \frac{2}{7} \geq b_2 > \alpha_d > \frac{1}{4}$ then $b_1 + b_2 \geq 1 - \alpha_d > 1 - b_2$ must be true, therefore

$$\begin{aligned}
&W(b_1) + W(b_2) = \\
&= \frac{6}{5p+1}b_1 + \frac{p-1}{(2p+1)(5p+1)} + \frac{18p}{(2p+1)(5p+1)}b_2 - \frac{p-1}{(2p+1)(5p+1)} = \\
&\quad = (b_1 + b_2)\frac{6}{5p+1} + b_2\frac{18p - 12p - 6}{(2p+1)(5p+1)} = \\
&\quad > (1 - b_2)\frac{6}{5p+1} + b_2\frac{6p-6}{(2p+1)(5p+1)} = \\
&\quad = \frac{6}{5p+1} - b_2\frac{6p+12}{(2p+1)(5p+1)} \geq \frac{3}{3p+1}
\end{aligned}$$

since $b_2 \leq \frac{2}{7}$.

If both paths are $\leq \frac{1}{3}$ then $\alpha_{d+1} > \alpha_d$.

If there are at least three paths, all $> \frac{1}{4}$, then

$$W(b_1) + W(b_2) + W(b_3) \geq \frac{3}{4}\frac{18p}{(2p+1)(5p+1)} - \frac{3p-3}{(2p+1)(5p+1)} > \frac{3}{3p+1}.$$

The other claim will be proved now, that the sum of the weights is at least $\frac{7}{3p+1}$ in the paired arms.

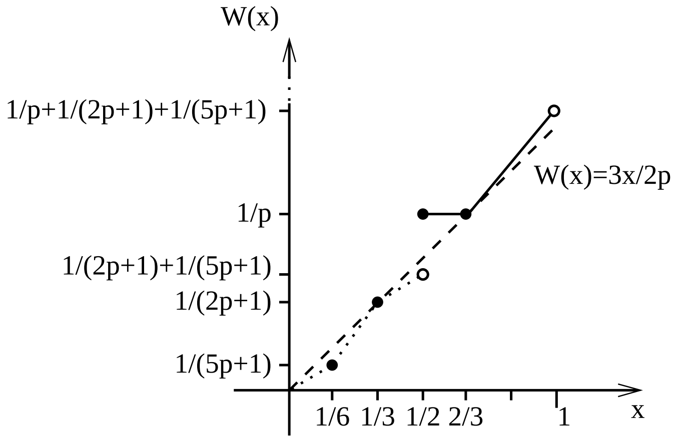

Fig. 2.6

The sum of the sizes must be at least $2-\alpha_d \geq \frac{12}{7}$ so there must be at least two paths in the paired arms since there is no path of size < 1 now.

Suppose there are exactly two paths. If both paths are $\geq \frac{1}{2}$ then

$$W(b_1)+W(b_2) \geq (b_1+b_2)\frac{3}{2p} > \frac{12}{7}\frac{3}{2p} \geq \frac{7}{3p+1} \tag{2.1}$$

since the weight function satisfies $W(x) \geq \frac{3}{2p}$ in $(\frac{1}{2},1]$ as the reader may see on Fig. 2.6.

If one path is $< \frac{1}{2}$ then the sum of the sizes is at most $\frac{3}{2} < \frac{12}{7}$ since the other path is < 1 in this case.

If there are 3 paths and each one is $\geq \frac{1}{2}$ then (2.1) also holds for 3 paths.

The next case is when $b_1 \geq b_2 \geq \frac{1}{2} > b_3 \geq \frac{1}{3}$, which implies $b_1+b_2 \geq (2-\frac{2}{7}-b_3)$. Hence we obtain

$$\sum_{i=1}^{3} W(b_i) \geq \frac{3}{2p}(b_1+b_2)+b_3\frac{6}{5p+1}+\frac{p-1}{(2p+1)(5p+1)} \geq$$
$$\geq \frac{3}{2p}\left(2-\frac{2}{7}-b_3\right)+b_3\frac{6}{5p+1}+\frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}.$$

since $b_3 \leq \frac{1}{2}$.

The same argument works with different coefficients for the case $b_1 \geq b_2 \geq \frac{1}{2} > \frac{1}{3} \geq b_3 > \frac{1}{4}$ and the case $b_1 \geq \frac{1}{2} > b_2 \geq b_3 \geq \frac{1}{3}$.

The next case is explained more detailed then it is necessary because there will be some other cases when this general method will be used again. If $b_1 \geq \frac{1}{2} > b_2 > \frac{1}{3} \geq b_3 > \frac{1}{4}$ then $b_1 \geq 2-\frac{2}{7}-b_2-b_3$ is obtained. Therefore

$$\sum_{i=1}^{3} W(b_i) \geq \frac{3}{2p}b_1+\frac{6}{5p+1}b_2+\frac{p-1}{(2p+1)(5p+1)}+$$
$$+\frac{18p}{(2p+1)(5p+1)}b_3-\frac{p-1}{(2p+1)(5p+1)} \geq$$
$$\geq \frac{3}{2p}\left(2-\frac{2}{7}-b_2-b_3\right)+\frac{6}{5p+1}b_2+\frac{18p}{(2p+1)(5p+1)}b_3 =$$
$$= \frac{18}{7p}-b_2\frac{3p+3}{2p(5p+1)}+b_3\frac{6p^2-21p-3}{2p(2p+1)(5p+1)}$$

holds. In general the coefficients of b_2 and b_3 may be independently positive or negative for some fixed p but the expression can reach its minimum only if b_2 and b_3 have minimum or maximum value. If we are interested in the minimum of the expression the following four cases must be checked:

1) $b_2 = \frac{1}{2} - \varepsilon$ and $b_3 = \frac{1}{3}$,
2) $b_2 = \frac{1}{2} - \varepsilon$ and $b_3 = \frac{1}{4} + \varepsilon$,
3) $b_2 = \frac{1}{3}\varepsilon$ and $b_3 = \frac{1}{3}$,
4) $b_2 = \frac{1}{3} + \varepsilon$ and $b_3 = \frac{1}{4} + \varepsilon$.

In each case an inequality is obtained with one variable and, though it may be quite complicated, it may be proved that the inequality is true. In fact the author used the computer software *MAPLE* to find the proofs faster.

In this particular case it is easy to see that the coefficient of b_2 is always negative and the coefficient of b_3 is positive if $p \geq 4$ so we need to check only case 2) for $p \geq 4$. Case 2):

$$\frac{18}{7p} - b_2\frac{3p+3}{2p\,(5p+1)} + b_3\frac{6p^2-21p-3}{2p\,(2p+1)\,(5p+1)} - \frac{7}{3p+1} = \frac{274p^3+859p^2+586p+81}{56p\,(5p+1)\,(2p+1)\,(3p+1)} > 0.$$

If $1 > b_1 \geq \frac{1}{2} > \frac{1}{3} > b_2 \geq b_3$ then $b_1 + b_2 + b_3 < \frac{5}{3} < \frac{12}{7}$ so the proof may be continued with the case of four paths.

When two of the four paths are $\geq \frac{1}{2}$ then we obtain

$$\sum_{i=1}^{4} W(b_i) \geq 2\frac{1}{p} + 2W\left(\frac{1}{4}\right) > \frac{7}{3p+1}$$

if $p \geq 4$, since the other two path are at least $\frac{1}{4}$.

The next case is $b_1 \geq \frac{1}{2} > b_2 \geq b_3 \geq b_4 > \frac{1}{3}$, when

$$\sum_{i=1}^{4} W(b_i) > \frac{1}{p} + 3\frac{1}{2p+1} \geq \frac{7}{3p+1}$$

holds.

If $b_1 > \frac{1}{2} \geq b_2 \geq b_3 > \frac{1}{3} \geq b_4 > \frac{1}{4}$ then one can apply the detailed method explained above and obtain

$$\begin{aligned}&\sum_{i=1}^{4} W(b_i)\\ &\geq \left(2 - \frac{2}{7} - b_2 - b_3 - b_4\right)\frac{3}{2p} + (b_2+b_3)\frac{6}{5p+1} + 2\frac{p-1}{(2p+1)\,(5p+1)} +\\ &+ b_4\frac{18p}{(2p+1)\,(5p+1)} - \frac{p-1}{(2p+1)\,(5p+1)} > \frac{7}{3p+1},\end{aligned}$$

because $b_2 + b_3$ may be considered as one variable.

The same method works with the appropriate coefficients if $b_1 \geq \frac{1}{2} > b_2 > \frac{1}{3} \geq b_3 \geq b_4 > \frac{1}{4}$.

If $b_1 \geq \frac{1}{2} > \frac{1}{3} \geq b_2 \geq b_3 \geq b_4 > \frac{1}{4}$ then

$$\begin{aligned}\sum_{i=1}^{4} W(b_i) &\geq \left(2 - \frac{2}{7} - b_2 - b_3 - b_4\right)\frac{3}{2p} +\\ &+ (b_2+b_3+b_4)\frac{18p}{(2p+1)\,(5p+1)} - 3\frac{p-1}{(2p+1)\,(5p+1)} > \frac{7}{3p+1}\end{aligned}$$

because the coefficient of $(b_2 + b_3 + b_4)$ will be positive if $p \geq 4$ and $b_2 + b_3 + b_4 > \frac{3}{4}$.

In the rest of this case all paths have length $< \frac{1}{2}$. If all paths are also $> \frac{1}{3}$ then

$$\sum_{i=1}^{4} W(b_i) \geq \left(2 - \frac{2}{7}\right) \frac{6}{5p+1} + 4\frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}$$

is obtained.

If $\frac{1}{2} > b_1 \geq b_2 \geq b_3 > \frac{1}{3} \geq b_4 > \frac{1}{4}$ then similarly to the previous cases

$$\sum_{i=1}^{4} W(b_i) \geq \left(2 - \frac{2}{7} - b_4\right) \frac{6}{5p+1} + 3\frac{p-1}{(2p+1)(5p+1)} + \\ + b_4\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}.$$

If $\frac{1}{2} > b_1 \geq b_2 > \frac{1}{3} \geq b_3 \geq b_4 > \frac{1}{4}$ then $2 - \frac{2}{7} \leq 2 - \alpha_d < \sum_{i=1}^{4} b_i > \frac{5}{3}$ is a contradiction. This was the last case of four paths so we can continue with five paths.

If one of the paths is $\leq \frac{1}{2}$ then we obtain

$$\sum_{i=1}^{5} W(b_i) \geq \frac{1}{p} + 4W\left(\frac{1}{4}\right) = \frac{1}{p} + \frac{2}{2p+1} + \frac{2}{5p+1} > \frac{7}{3p+1}$$

is obtained because all the other paths $> \frac{1}{4}$.

If $\frac{1}{2} > b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5 > \frac{1}{3}$ then

$$\sum_{i=1}^{5} W(b_i) > 5W\left(\frac{1}{2}\right) > \frac{7}{3p+1}$$

is obtained.

If $\frac{1}{2} > b_1 \geq b_2 \geq b_3 \geq b_4 > \frac{1}{3} \geq b_5 > \frac{1}{4}$ then

$$\sum_{i=1}^{5} W(b_i) \geq \left(2 - \frac{2}{7} - b_5\right) \frac{6}{5p+1} + b_5\frac{18p}{(2p+1)(5p+1)} + 3\frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}$$

holds because the coefficient of b_5 is positive if $p \geq 4$ and $b_5 > \frac{1}{4}$.

If $\frac{1}{2} > b_1 \geq b_2 \geq b_3 > \frac{1}{3} \geq b_4 \geq b_5 > \frac{1}{4}$ then the proof is the same as in the previous case.

If $\frac{1}{2} > b_1 \geq b_2 > \frac{1}{3} \geq b_3 \geq b_4 \geq b_5 > \frac{1}{4}$ then

$$\sum_{i=1}^{5} W(b_i) \geq (b_1+b_2)\frac{6}{5p+1} + \left(2 - \frac{2}{7} - b_1 - b_2\right) \frac{18p}{(2p+1)(5p+1)} + \frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}$$

hold because the coefficient of $b_1 + b_2$ is negative if $p \geq 4$ and $b_1 + b_2 < 1$.

If $\frac{1}{2} > b_1 > \frac{1}{3} \geq b_2 \geq b_3 \geq b_4 \geq b_5 > \frac{1}{4}$ then

$$\sum_{i=1}^{5} W(b_i) \geq b_1\frac{6}{5p+1} + \left(2 - \frac{2}{7} - b_1\right) \frac{18p}{(2p+1)(5p+1)} + 3\frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}$$

is obtained since the coefficient of b_1 is negative if $p \geq 4$ and $b_1 < \frac{1}{2}$.

If all 5 paths are $< \frac{1}{3}$ then the sum of their sizes is $< \frac{5}{3} < \frac{12}{7}$ which is a contradiction therefore we continue with the case of six paths.

If one path is $\geq \frac{1}{2}$ then

$$\sum_{i=1}^{6} W(b_i) \geq \frac{1}{p} + 5W\left(\frac{1}{4}\right) > \frac{7}{3p+1}$$

holds since the other paths are $> \frac{1}{4}$.

The proof is similar if $\frac{1}{2} > b_1 \geq b_2 > \frac{1}{3}$ since $p \geq 4$.

If $\frac{1}{2} > b_1 > \frac{1}{3} \geq b_2 \geq \ldots \geq b_6 > \frac{1}{4}$ then

$$\sum_{i=1}^{6} W(b_i) > b_1 \frac{6}{5p+1} + \frac{p-1}{(2p+1)(5p+1)} + \\ + \left(2 - \frac{2}{7} - b_1\right) \frac{18p}{(2p+1)(5p+1)} - 5\frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}$$

since $b_1 < \frac{1}{2}$.

The last case is when there are seven or more paths. All of them must be over $\frac{1}{4}$ thus

$$\sum_{i=1}^{t} W(b_i) > 7W\left(\frac{1}{4}\right) > \frac{7}{3p+1}.$$

To complete the proof of this part a contradiction is shown. It is supposed that$\alpha_{d+1} \leq \alpha_d$ which implies that the sum of the weights is $\geq \frac{7}{3p+1}$ in the paired arms and the weight sum is $\geq \frac{3}{3p+1}$ in each of the $p-2$ single arm. This is a contradiction since

$$\frac{7}{3p+1} + (p-2)\frac{3}{3p+1} = 1.$$

Case 5: $\frac{2}{7} < \alpha_d \leq \frac{1}{3}$, the center is covered.

Suppose that $\alpha_{d+1} \not> \alpha_d$. It will be shown that this implies $\sum_{i=1}^{t} W(b_i) \geq 1$, a contradiction. Unfortunately we have to go through a number of cases again but as α_d increases, the number of the cases decreases.

There is an exceptional case which cannot be covered by the usual method so it have to be proved in a different way. This is the case when there are 5 paths in the paired arms (the center is covered) and $\frac{1}{2} > b_1 \geq b_2 > \frac{1}{3} \geq b_3 \geq b_4 \geq b_5 > \alpha_d > \frac{2}{7}$.

Let us prove this case first. It will be proved that the weight sum is at least

$$\frac{23p+13}{(2p+1)(5p+1)} + \alpha_d \frac{6p-24}{(2p+1)(5p+1)},$$

in the paired arms and the weight sums are at least

$$\frac{10p^2 - 16p - 12 - 6\alpha_d p + 24\alpha_d}{(2p+1)(5p+1)(p-2)}$$

in the single arms. The total weight sum is the sum of the weights in the paired arms and in the $(p-2)$ single arms:

$$\frac{23p+13}{(2p+1)(5p+1)} + \alpha_d \frac{6p-24}{(2p+1)(5p+1)} + \frac{10p^2-16p-12-6\alpha_d p+24\alpha_d}{(2p+1)(5p+1)} \geq 1,$$

which is a contradiction.

It is easy to prove the first claim. Now $b_1 + b_2 \geq 2 - b_3 - b_4 - b_5 - \alpha_d$ holds which implies

$$\begin{aligned}\sum_{i=1}^{5} W(b_i) &\geq (2-b_3-b_4-b_5-\alpha_d)\,\frac{6}{5p+1}+\\ &+ (b_3+b_4+b_5)\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} =\\ &= (b_3+b_4+b_5)\frac{6p-6}{(2p+1)(5p+1)} + \frac{12-6\alpha_d}{5p+1} - \frac{p-1}{(2p+1)(5p+1)} >\\ &> \alpha_d\frac{18p-18}{(2p+1)(5p+1)} + \frac{12-6\alpha_d}{5p+1} - \frac{p-1}{(2p+1)(5p+1)} =\\ &= \frac{23p+13}{(2p+1)(5p+1)} + \alpha_d\frac{6p-24}{(2p+1)(5p+1)}.\end{aligned}$$

The second claim about the single arms is proved next.

If there is only one path in the arm it must be over $\frac{2}{3}$ otherwise $\alpha_{d+1} > \alpha_d$. The weight of a path over $\frac{2}{3}$ is at least $\frac{1}{p}$, and the inequality

$$\frac{1}{p} > \frac{10p^2-16p-12-6\alpha_d p+24\alpha_d}{(2p+1)(5p+1)(p-2)}$$

holds for $\alpha_d \leq \frac{1}{3}$, proving our claim.

If there are two paths, $b_1 \geq b_2 > \frac{1}{3}$ then

$$\begin{aligned}&W(b_1)+W(b_2) \geq\\ &> (1-\alpha_d)\frac{6}{5p+1} + 2\frac{p-1}{(2p+1)(5p+1)} \geq \frac{10p^2-16p-12-6\alpha_d p+24\alpha_d}{(2p+1)(5p+1)(p-2)}\end{aligned}$$

holds, since $\alpha_d \leq \frac{1}{3}$.

If $b_1 > \frac{1}{3} \geq b_2 > \frac{2}{7}$ then

$$\begin{aligned}W(b_1)+W(b_2) &\geq b_1\frac{6}{5p+1} + \frac{p-1}{(2p+1)(5p+1)}+\\ &+ b_2\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} >\\ &> (1-b_2-\alpha_d)\,\frac{6}{5p+1} + b_2\frac{18p}{(2p+1)(5p+1)} =\\ &= b_2\frac{6p-6}{(2p+1)(5p+1)} + \frac{6-6\alpha_d}{5p+1} \geq\\ &\geq \frac{10p^2-16p-12-6\alpha_d p+24\alpha_d}{(2p+1)(5p+1)(p-2)}\end{aligned}$$

is obtained.

$\frac{1}{3} \geq b_1 \geq b_2$ is not possible so the only case left is when there are at least 3 paths. The weight of each path is at least $W(\alpha_d)$ and

$$3W(\alpha_d) \geq \frac{10p^2 - 16p - 12 - 6\alpha_d p + 24\alpha_d}{(2p+1)(5p+1)(p-2)}$$

can be proved easily.

The proof of the exceptional case is completed. Notice that

$$\frac{10p^2 - 16p - 12 - 6\alpha_d p + 24\alpha_d}{(2p+1)(5p+1)(p-2)} \geq \frac{2}{2p+1}$$

holds for the previous lower bound if $p \geq 4$ and $\alpha_d \leq \frac{1}{3}$. Thus it can be proved that the sum of the weights is at least $\frac{2}{2p+1}$ in the single arms since these inequalities does not depend on the conditions of the exceptional case.

In order to finish this case it will be proved that the weight sum is at least $\frac{5}{2p+1}$ in the paired arms (except the exceptional case).

It is easy to see that there are least 2 paths otherwise the weight of the only path is at least 1. If both paths are $\geq \frac{1}{2}$ then the lower bound $\frac{3}{2p}x$ on the weight of a path of length x can be used as earlier. The other observation is that $b_1 + b_2 \geq 2 - \alpha_d \geq \frac{5}{3}$. Thus

$$W(b_1) + W(b_2) \geq \frac{3}{2p}(b_1 + b_2) \geq \frac{3}{2p}\frac{5}{3} > \frac{5}{2p+1}$$

holds. The same proof can be applied if there are more than 2 paths and all are $\geq \frac{1}{2}$. In the rest of the proof there will be at least one path which is not larger than $\frac{1}{2}$. Therefore if there are only two paths then $\alpha_{d+1} > \alpha_d$, since the other path is at most 1.

The cases when there are 3 paths in the paired arms are proved next. If $b_1 \geq b_2 \geq \frac{1}{2} > b_3 \leq \frac{1}{3}$ then

$$\sum_{i=1}^{3} W(b_i) \geq 2\frac{1}{p} + \frac{1}{2p+1} > \frac{5}{2p+1}$$

holds.

If $b_1 \geq b_2 \geq \frac{1}{2} \geq \frac{1}{3} > b_3 > \alpha_d > \frac{2}{7}$ then a better lower bound must be used for the paths $\geq \frac{1}{2}$. It is obvious that

$$W(x) \geq x\frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2 - 3p - 1}{p(2p+1)(5p+1)}$$

in $\left[\frac{1}{2}, 1\right)$. Using the bound $b_1 + b_2 > 2 - \alpha_d - b_3$

$$\begin{aligned}\sum_{i=1}^{3} W(b_i) \geq (b_1+b_2)\frac{21p+6}{(2p+1)(5p+1)} - 2\frac{4p^2-3p-1}{p(2p+1)(5p+1)} + \\ + b_3\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} \geq \\ \geq (2 - \alpha_d - b_3)\frac{21p+6}{(2p+1)(5p+1)} + \\ + b_3\frac{18p}{(2p+1)(5p+1)} - \frac{9p^2-7p-2}{p(2p+1)(5p+1)} = \\ = -b_3\frac{3p+6}{(2p+1)(5p+1)} + \frac{13p+2}{p(5p+1)} \geq \frac{5}{2p+1}\end{aligned}$$

is obtained since now $b_3 < \frac{1}{3}$ and $\alpha_d \le \frac{1}{3}$.

If $b_1 \ge \frac{1}{2} > b_2 \ge b_3 \ge \frac{1}{3}$ then none of our usual methods works. However, if we take into consideration that these paths got their place in the polyp with the First Fit algorithm then we can settle it easily. The following more general claim is proved because this method is used in other cases, too.

Claim 2.4 *If the paired arms contain the paths $b_1, \dots, b_t$, for every $1 \le i \le t$ and every partition of $\{b_1, \dots, b_t\} - \{b_i\}$ into two sets (one of them may be empty) the sum of the sizes in one of the parts is $< 1 - b_i$ then the center cannot be covered.*

Proof of Claim 2.4: The only reason to cover the center with b_i is that it does not fit to neither arm of the paired arms. This happens only if the sum of the sizes in both of these arms is $\ge 1 - b_i$. ■

If $b_1 \ge \frac{1}{2} > b_2 \ge b_3 \ge \frac{1}{3}$ ten the conditions of Claim 2.4 clearly holds, the center cannot be covered.

If $b_1 \ge \frac{1}{2} > b_2 \ge \frac{1}{3} > b_3 > \frac{2}{7}$ then

$$\begin{aligned}\sum_{i=1}^{3} W(b_i) \ge\ & b_1 \frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} + \\ & + b_2 \frac{6}{5p+1} + \frac{p-1}{(2p+1)(5p+1)} + \\ & + b_3 \frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} \ge \\ \ge\ & \left(2 - \frac{1}{3} - b_2 - b_3\right) \frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} + \\ & + b_2 \frac{6}{5p+1} + b_3 \frac{18p}{(2p+1)(5p+1)} > \frac{5}{2p+1}\end{aligned}$$

using the method explained in detail earlier.

If $b_1 \ge \frac{1}{2} \ge \frac{1}{3} > b_2 \ge b_3 > \frac{2}{7}$ then

$$\begin{aligned}\sum_{i=1}^{3} W(b_i) \ge\ & b_1 \frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} + \\ & + (b_2+b_3) \frac{18p}{(2p+1)(5p+1)} - 2\frac{p-1}{(2p+1)(5p+1)} \ge \\ \ge\ & \left(2 - \frac{1}{3} - b_2 - b_3\right) \frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} + \\ & + (b_2+b_3) \frac{18p}{(2p+1)(5p+1)} - 2\frac{p-1}{(2p+1)(5p+1)} > \frac{5}{2p+1}\end{aligned}$$

since the coefficient of $b_2 + b_3$ is negative and $b_2 + b_3 \le \frac{2}{3}$.

If there is no path $\ge \frac{1}{2}$ then the sum of the sizes may reach only $\frac{3}{2}$ but then $\alpha_{d+1} > \alpha_d$, so we can continue with the case of four paths in the paired arms.

If there are at least two paths $\ge \frac{1}{2}$ then, since the other two must be $> \frac{2}{7}$, we obtain

$$\sum_{i=1}^{3} W(b_i) \ge \frac{2}{p} + 2W\left(\frac{2}{7}\right) > \frac{5}{2p+1}.$$

If $b_1 \geq \frac{1}{2} > b_2 \geq b_3 \geq b_4 \geq \frac{1}{3}$ then

$$\sum_{i=1}^{4} W(b_i) \geq \frac{1}{p} + 3\frac{1}{2p+1} > \frac{5}{2p+1}$$

holds.

If $b_1 > \frac{2}{3} > \frac{1}{2} > b_2 \geq b_3 \geq \frac{1}{3} > b_4 > \frac{2}{7}$ then the complicated method again must be used again:

$$\begin{aligned}\sum_{i=1}^{4} W(b_i) \geq\ & b_1\frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} + (b_2+b_3)\frac{6}{5p+1} + \\ & + 2\frac{p-1}{(2p+1)(5p+1)} + b_4\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} > \\ & > b_1\frac{21p+6}{(2p+1)(5p+1)} + (2-\alpha_d-b_1-b_4)\frac{6}{5p+1} + \\ & + b_4\frac{18p}{(2p+1)(5p+1)} - \frac{3p^2-2p-1}{p(2p+1)(5p+1)} > \frac{5}{2p+1}\end{aligned}$$

holds since $\alpha_d < b_4$, the coefficient of b_1 is positive, the coefficient of b_4 is negative, $b_1 > \frac{2}{3}$ and $b_4 < \frac{1}{3}$.

If $\frac{2}{3} \geq b_1 \geq \frac{1}{2} > b_2 \geq b_3 \geq \frac{1}{3} > b_4 > \frac{2}{7}$ then

$$\begin{aligned}\sum_{i=1}^{4} W(b_i) \geq\ & \frac{1}{p} + \left(2-\frac{2}{3}-b_4\right)\frac{6}{5p+1} + 2\frac{p-1}{(2p+1)(5p+1)} + \\ & + b_4\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} > \frac{5}{2p+1}\end{aligned}$$

since $b_1 \leq \frac{2}{3}$, the coefficient of b_4 is positive and $b_4 > \frac{2}{7}$.

If $\frac{2}{3} \geq b_1 \geq \frac{1}{2} > b_2 \geq \frac{1}{3} > b_3 \geq b_4 > \frac{2}{7}$ then

$$\begin{aligned}\sum_{i=1}^{4} W(b_i) \geq\ & \frac{1}{p} + \left(2-\frac{2}{3}-b_3-b_4\right)\frac{6}{5p+1} + \frac{p-1}{(2p+1)(5p+1)} + \\ & + (b_3+b_4)\frac{18p}{(2p+1)(5p+1)} - 2\frac{p-1}{(2p+1)(5p+1)} > \frac{5}{2p+1}\end{aligned}$$

because $b_1 \leq \frac{2}{3}$, the coefficient of b_3+b_4 is positive and $b_3+b_4 > \frac{4}{7}$.

If $\frac{2}{3} \geq b_1 \geq \frac{1}{2} \geq \frac{1}{3} > b_2 \geq b_3 \geq b_4 > \frac{2}{7}$ then it is easy to see that $\alpha_{d+1} > \alpha_d$.

If $\frac{1}{2} > b_1 \geq b_2 \geq b_3 \geq b_4$ then the center cannot be covered by Claim 2.4. This completes the case of 4 paths.

In the next cases there are five paths in the paired arms. If at least one of them is $\geq \frac{1}{2}$ then

$$\sum_{i=1}^{5} W(b_i) > \frac{1}{p} + 4W\left(\frac{2}{7}\right) > \frac{5}{2p+1}.$$

is a good lower bound, since the other paths are over $\frac{2}{7}$.

If all paths are $\geq \frac{1}{3}$ then the weight sum is clearly $\geq \frac{5}{2p+1}$.

If $\frac{1}{2} \geq b_1 \geq b_2 \geq b_3 \geq b_4 \geq \frac{1}{3} > b_5 > \alpha_d > \frac{2}{7}$ then the usual method works, thus we obtain

$$\begin{aligned}\sum_{i=1}^{5} W(b_i) &\geq (b_1+b_2+b_3+b_4)\frac{6}{5p+1} + 4\frac{p-1}{(2p+1)(5p+1)} + \\ &+ b_5\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} \geq \\ &\geq (2-\alpha_d-b_5)\frac{6}{5p+1} + b_5\frac{18p}{(2p+1)(5p+1)} + 3\frac{p-1}{(2p+1)(5p+1)} > \frac{5}{2p+1},\end{aligned}$$

since $\alpha_d < b_5$. The same idea works if $\frac{1}{2} \geq b_1 \geq b_2 \geq b_3 \geq \frac{1}{3} > b_4 \geq b_5 > \alpha_d > \frac{2}{7}$.

The next case, $\frac{1}{2} > b_1 \geq b_2 \geq \frac{1}{3} > b_3 \geq b_4 \geq b_5 > \alpha_d > \frac{2}{7}$, is the exceptional case which is already proved.

If $\frac{1}{2} \geq b_1 \geq \frac{1}{3} > b_2 \geq b_3 \geq b_4 \geq b_5 > \alpha_d > \frac{2}{7}$ then the center remains uncovered. By Claim 2.4 the only way the center could be covered if two of the paths $< \frac{1}{3}$ are in one of the paired arms, the other two are in the other side and the path of size $\geq \frac{1}{2}$ covers the center. However, this cannot occur since any 3 of the 4 paths of size $< \frac{1}{3}$ fits completely to one arm thus we do not place a path of size $< \frac{1}{3}$ into a new arm until there are three of them in the first arm.

The last case is when there are at least 6 paths in the paired arms. However, it is easy to prove this, because

$$\sum_{i=1}^{t} W(b_i) \geq 6W\left(\frac{2}{7}\right) > \frac{5}{2p+1}.$$

Case 6: $\frac{1}{3} < \alpha_d < \frac{1}{2}$, the center is not covered.

As usual, the case when the center is not covered must be distinguished. If there are two arms where the empty space is more than $\frac{\alpha_d}{2}$ then a path of size α_d fits to this place. Hence it may be supposed that there is only one arm — A_1 — where the sum of the sizes is $1 - \frac{\alpha_d}{2} - z$ such that $0 < z \leq \frac{\alpha_d}{2}$. If $z > \frac{\alpha_d}{2}$ then a paths of size α_d already fits to this arm. In this way the sum of the sizes in all other arms is $\geq 1 - \frac{\alpha_d}{2} + z$. It will be proved that if the sum of the sizes in an arm is $S > \frac{1}{2}$ then the sum of the weights is

$$\geq \frac{6}{5p+1}S + \frac{2p-2}{(2p+1)(5p+1)}.$$

If there is only one path and $S > \frac{2}{3}$ then

$$W(S) = \frac{(21p+6)}{(2p+1)(5p+1)}S - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} > \frac{6}{5p+1}S + \frac{2p-2}{(2p+1)(5p+1)}$$

holds.

If there is only one path and $\frac{1}{2} < S < \frac{2}{3}$ then

$$W(S) = \frac{1}{p} > \frac{6}{5p+1}\frac{2}{3} + \frac{2p-2}{(2p+1)(5p+1)} \geq \frac{6}{5p+1}S + \frac{2p-2}{(2p+1)(5p+1)}.$$

If there are two paths then the claim is true, because

$$W(x) \geq \frac{6}{5p+1}x + \frac{p-1}{(2p+1)(5p+1)}$$

holds in the interval $\left[\frac{1}{3}, 1\right)$.

Therefore the proof is complete if $\alpha_d \leq \frac{2}{5}$, because

$$\sum_{i=1}^{t} W(b_i) \geq$$
$$\geq \left[\left(1 - \frac{\alpha_d}{2} + z\right)(p-1) + \left(1 - \frac{\alpha_d}{2} - z\right)\right] \frac{6}{5p+1} + p\frac{2p-2}{(2p+1)(5p+1)} >$$
$$> p\left(1 - \frac{\alpha_d}{2}\right) \frac{6}{5p+1} + p\frac{2p-2}{(2p+1)(5p+1)} > 1.$$

holds in this case.

If $\frac{2}{5} < \alpha_d < \frac{1}{2}$ then it can be shown that the sum of the sizes in each arm is at least $\frac{1}{p}$. If there is a path $\geq \frac{1}{2}$ in the arm then this is clearly true. If there is only one path but it is $< \frac{1}{2}$ then the empty space is larger than α_d which is a contradiction. If there are two paths then both must be larger than $\alpha_d > \frac{2}{5}$ thus

$$W(b_1) + W(b_2) > \frac{4}{5}\frac{6}{5p+1} + 2\frac{p-1}{(2p+1)(5p+1)} > \frac{1}{p}.$$

Therefore the weight sum is at least $\frac{1}{p}$ in each arm thus the weight sum in the polyp is at least 1, proving our claim.

Case 7: $\frac{1}{3} < \alpha_d \leq \frac{5}{12}$, the center is covered.

Like in the previous cases a lower bound will be proved for the paired arms and an other lower bound for the other arms. It will be proved that the weight sum is $> \frac{5}{2p+1}$ in the paired arms and $> \frac{2}{2p+1}$ in the other arms.

There are only two subcases in the latter case so we begin with the single arm. If there is only one path in the arm, b_1, then $b_1 \geq 1 - \alpha_d \geq 1 - \frac{5}{12}$ otherwise $\alpha_{d+1} > \alpha_d$, so clearly $W(b_1) \geq \frac{1}{p} > \frac{2}{2p+1}$.

If there are two paths then both must be $> \frac{1}{3}$ assuring that the weight sum is $> \frac{2}{2p+1}$. Finally there is not enough place for three paths each $> \frac{1}{3}$.

Now the other claim is proved for the paired arms. If there is only one path in the paired arms then $\alpha_{d+1} > \alpha_d$or the weight sum is ≥ 1. If there are two paths then it may be supposed that both are < 1. However, it is easy to see that the center cannot be covered in this case.

We may suppose new that there are three paths. If all three are $\geq \frac{1}{2}$ then we obtain

$$\sum W(b_i) > \frac{3}{p} > \frac{5}{2p+1}.$$

If $b_1 \geq b_2 \geq \frac{1}{2} > b_3 > \frac{1}{3}$ then

$$\sum W(b_i) \geq 2\frac{1}{p} + \frac{1}{2p+1} > \frac{5}{2p+1}$$

holds.

The next case is $b_1 \geq \frac{1}{2} > b_2 \geq b_3 > \alpha_d > \frac{1}{3}$. In this case the center cannot be covered again by Claim 2.4.

There are no more cases with three paths in the paired arms, since if there are three paths each $< \frac{1}{2}$ then $\alpha_{d+1} > \alpha_d$.

There are two more cases when there are four paths but at most one is $\geq \frac{1}{2}$. $b_1 \geq \frac{1}{2} > b_2 \geq b_3 \geq b_4 > \alpha_d > \frac{1}{3}$. In this case

$$\sum_{i=1}^{4} W(b_i) \geq \frac{1}{p} + 3\frac{1}{2p+1} > \frac{5}{2p+1}$$

holds. If $\frac{1}{2} > b_1 \geq b_2 \geq b_3 \geq b_4 > \alpha_d > \frac{1}{3}$ then the center cannot be covered by Claim 2.4.

Therefore there is only one case left, when there are at least five paths in the paired arms. Now we obtain

$$\sum_{i=1}^{t} W(b_i) \geq 5\frac{1}{2p+1}$$

immediately completing the proof for this set of cases.

Case 8: $\frac{5}{12} < \alpha_d < \frac{1}{2}$, the center is covered.

It will be proved in this case that the lower bound for the weight sum is $\frac{2}{p}$ in the paired arms and $\frac{1}{p}$ in the single arms. If there is only one path in a single arm then its weight is at least $\frac{1}{p}$, if there are at least two paths then

$$W(b_1) + W(b_2) \geq 2W\left(\frac{5}{12}\right) = \frac{12p+3}{(2p+1)(5p+1)} > \frac{1}{p}$$

holds.

It is clear that two paths in the paired arms cannot cover the center with the First Fit algorithm. If there are three paths but one of them is $\geq \frac{1}{2}$ then, similarly to the previous case,

$$W(b_1) + W(b_2) + W(b_3) \geq \frac{1}{p} + 2W\left(\frac{5}{12}\right) > \frac{2}{p}$$

is obtained. If all three paths are $< \frac{1}{2}$ then the center cannot be covered by Claim 2.4.

If there are at least four paths in the paired arms then

$$\sum_{i=1}^{t} W(b_i) \geq 4W\left(\frac{5}{12}\right) > \frac{2}{p}$$

holds.

Case 9: $\frac{1}{2} \leq \alpha_d < 1$.

Now $W(b_1) \geq \ldots \geq W(b_t) \geq \frac{1}{p}$ so if $t \geq p$ then $\sum_{i=1}^{t} W(b_i) \geq 1$ holds. Suppose that there are less than p paths. From the definition of the algorithm it is clear that in this way these paths must be in the ends of the arms thus the center is not covered. But if there are less than p paths then one of the arms is empty so a path of size ≥ 1 fits in, thus $\alpha_{d+1} > \alpha_d$. This completes the proof of Lemma 2.3. ∎

LEMMA 2.5 *Let a polyp P_d of coarseness $\alpha_d < 1$ be filled with paths $b_1 \geq b_2 \geq \ldots \geq b_t$ in the completed First Fit algorithm. If $\sum_{i=1}^{t} W(b_i) = 1 - \beta_d$ $(0 < \beta_d < 1)$ then either $\alpha_{d+1} \geq \min(\alpha_d + \frac{1}{3}\beta_d, 1)$ holds or there is an arm containing at most one path of size $< \frac{1}{2}$.*

PROOF: The second alternative is satisfied unless there are at least two paths or one with size $\leq \frac{1}{2}$ in every arm. Suppose that $\alpha_{d+1} < \alpha_d + \frac{1}{3}\beta_d < 1$ when the list of paths L is packed to the polyps with the First Fit algorithm. A modified list L' will be created in the following way. Some of the paths in P_d will be enlarged. The proof is similar if $\alpha_d + \frac{1}{3}\beta_d \geq 1$.

Case 1: The center is covered.

Let A_1 and A_2 denote the paired arms and b_c the path which covers the center. The empty part is "together" in this pair of arms, because the First Fit algorithm was used to pack the paths. Let e denote the length of the empty part. This empty part does not contain the center, so it is contained by an arm completely, suppose that it is in A_1. A_1 contains a part of b_c and it must contain some paths which are as far from the center as possible. Let b_e be the closest path to the center among these paths. If $b_e \geq \frac{1}{2}$ then this path will be enlarged, let $b'_e = b_e + \min(\frac{1}{3}\beta_d, e)$. In this way the size of this empty part decreases by $\frac{1}{3}\beta_d$ or is becomes 0. If $b_e < \frac{1}{2}$ then other paths will be enlarged in the following way. Suppose that there is another path, b_f, in A_1 besides b_e. If $b_f \geq \frac{1}{2}$ then let $b'_f = b_f + \min(\frac{1}{3}\beta_d, e)$. If $b_f < \frac{1}{2}$ then let $b'_e = b_e + x$ and $b'_f = b_f + y$ such that $x + y = \min(\frac{1}{3}\beta_d, e)$, $x < \frac{1}{2} - b_e$ and $y < \frac{1}{2} - b_f$. It is easy to see that such x, y exist since $\min(\frac{1}{3}\beta_d, e) < 1 - b_e - b_f$.

There is one more possibility that $b_c < \frac{1}{2}$, A_1 contains a part of b_c but there is no other path in it. If $b_e \geq \frac{1}{2}$ then let $b'_e = b_e + \min(\frac{1}{3}\beta_d, e)$. If $b_c \geq \frac{1}{2}$ then $b'_c = b_c + \min(\frac{1}{3}\beta_d, e)$. If $b_c < \frac{1}{2}$ and $b_e < \frac{1}{2}$ then let b_f a path from A_2. If $b_f \geq \frac{1}{2}$ then let $b'_e = b_e + x$, $b'_c = b_c + y$ and $b'_f = b_f + z$ such that $x + y + z = \min(\frac{1}{3}\beta_d, e)$, $x < \frac{1}{2} - b_e$, $y < \frac{1}{2} - b_c$ and $z < 1 - b_f$. It is easy to see that such x, y, z exist since $\min(\frac{1}{3}\beta_d, e) < 2 - b_e - b_c - b_f$. If $b_f < \frac{1}{2}$ then there chose a further path, b_g, from A_2. It may be supposed that it is $< \frac{1}{2}$. Now enlarge b_e, b_c, b_f, b_g similarly as above.

In each case the empty part decrease by $\frac{1}{3}\beta_d$ or with e such that if a path $b_x < \frac{1}{2}$ then $b'_x < \frac{1}{2}$, if $b_x < 1$ then $b'_x < 1$ and the sum of the enlarged paths in A_2 is < 1. The last condition ensures that if L' is packed with the First Fit algorithm then b'_c covers the center.

It is easy to see that such an enlargement can be done in the unpaired arms in the same way.

Case 2: The center is not covered.

The empty part in these arms are attached to the center and all paths in the arms are as far from the center as possible. (There must be at least one path in each arm.) Let $e_1 \geq e_2 \geq \ldots \geq e_k$ be the sizes of the empty parts in the unpaired arms, b_{e_i} the closest path to the center in arm A_i. By our assumption $\alpha_d + \frac{1}{3}\beta_d > e_1 + e_2$. If $e_1 < \alpha_d + \frac{1}{6}\beta_d$ then let $b'_{e_i} = b_{e_i} + \min(\frac{1}{6}\beta_d, e_i)$ for all i. If $e_1 \geq \alpha_d + \frac{1}{6}\beta_d$ then $\frac{1}{6}\beta_d > e_2 \geq e_3 \geq \ldots$ holds since $e_1 + e_2 < \alpha_d + \frac{1}{3}\beta_d$. Let $b'_{e_1} = b_{e_1} + \frac{1}{3}\beta_d - e_2 + \varepsilon$ and $b'_{e_i} = b_{e_i} + e_i - \varepsilon$ for $i = 2, 3, \ldots, k$ where ε is as small as possible. If $b_{e_i} < \frac{1}{2}$ for some i then there must be at another path $b_{e'_i} < \frac{1}{2}$ in the same arm where b_i is. Then it is easy to see that it is possible to enlarge b_i and $b_{i'}$ together with the same size, such that $b'_i < \frac{1}{2}$ and $b'_{i'} < \frac{1}{2}$. This completes Case 2.

The modified list $L' = b'_1, b'_2, \ldots$ is obtained by replacing the above paths by the enlarged paths. If L' is packed with the First Fit algorithm then every paths will go the same arm of the same polyp in the same order as the corresponding path in L, because no path fits to an earlier place, but every path fits to its "old" place. On the other hand it is easy to see that after packing L' there will not be enough empty space to insert an additional path of size $> \alpha_d$

The slope of W in the interval $[0, \frac{1}{2})$ and $\left[\frac{1}{2}, 1\right)$ (we did not "jump" over $\frac{1}{2}$ during the enlargements) is less than $\frac{3}{p}$, the sum of the enlargements is $\leq (p-1)\frac{1}{3}\beta_d$ therefore

$$\sum W(b'_i) \leq \sum W(b_i) + (p-1)\frac{3}{p}\frac{1}{3}\beta_d = 1 - \frac{\beta_d}{p} < 1$$

is obtained. Thus L' would be a counter example to Lemma 2.3 which is a contradiction. ■

Now we are prepared to complete the proof of Theorem 2.1.

PROOF OF THEOREM 2.1: Let $Q_1, \ldots, Q_f$ denote the polyps for which $\sum_{i=1}^{t} W(b_j) = 1 - \beta_i$ with $\beta_i > 0$ in the First Fit algorithm in the same order as they appear in the list of the polyps. The other polyps have weight sum at least 1. Furthermore let γ_i be the coarseness of Q_i. $\gamma_i < 1$ holds for $i = 1, \ldots, f$, since Q_i cannot contain a path of size ≥ 1.

By Lemma 2.5 and the definition of coarseness,

$$\gamma_{i+1} > \gamma_i + \frac{1}{3}\beta_i \quad \text{for} \quad i = 1, \ldots, f-1,$$

except when Q_i has an arm containing at most one single path of size $< \frac{1}{2}$. Suppose Q_j has an arm containing one path of size $< \frac{1}{2}$ then a path of size $\geq \frac{1}{2}$ would fit to Q_j thus all the paths in the larger indexed polyp must be $\geq \frac{1}{2}$. Thus for $i > j$ Q_i cannot contain have an arm which contains one path of size $< \frac{1}{2}$. However, it is possible that Q_k for some $k > j$ has an empty arm. Then all the larger indexed polyps must have paths only of size > 1 which implies that their weight sum is not less than 1. Therefore if such k exists then $k = f$. So

$$\beta_i < 3(\gamma_{i+1} - \gamma_i)$$

holds for $i = 1, \ldots, j-1, j+1, \ldots, f-1$. Obviously $\beta_k, \beta_f < 1$ and it was also shown that $\gamma_1 < \ldots < \gamma_f < 1$ thus

$$\sum_{i=1}^{f} \beta_i < 3\sum_{i=1}^{j-1}(\gamma_{i+1} - \gamma_i) + 3\sum_{i=j+1}^{f-1}(\gamma_{i+1} - \gamma_i) + 2 < 3\sum_{i=1}^{f-1}(\gamma_{i+1} - \gamma_i) + 2 = 3(\gamma_f - \gamma_1) + 2 \leq 5.$$

Summing the weights for all paths in L

$$FF_p(L) - 5 \leq \sum W(l_i)$$

is obtained. Finally, applying Lemma 2.2 we conclude

$$FF_p(L) \leq \sum_{i=1}^{m} W(l_i) + 5 \leq OPT_p(L)\left[\frac{p-1}{5p+1} + \frac{p-1}{2p+1} + \frac{p-1}{p} + 1\right] + 5$$

completing the proof. ■

CONJECTURE 2.6 *If the arm size of the polyp is suitably large then there exists a list L for any large enough $OPT_p(L)$ such that for*

$$FF_p(L) \geq OPT_p(L)\left[\frac{p-1}{5p+1} + \frac{p-1}{2p+1} + \frac{p-1}{p+1} + 1\right] - const.$$

Note that this ratio is very close to the bound in Theorem 2.1. The difference is only $\frac{1}{p(p+1)}$. For some values of p this would give the following lower bounds for the ratio (the upper bounds are given by Theorem 2.1):

$$\begin{aligned} 2.076 &< R_4(< 2.227) \\ 2.184 &< R_5(< 2.318) \\ 2.260 &< R_6(< 2.380) \\ 2.522 &< R_{15}(< 2.570) \\ 2.7 &= \lim_{i\to\infty} R_i \end{aligned}$$

The following method may give a proof for this conjecture, however, the proof may be very long and complicated.

As it was mentioned before the arm size of the polyp will be 1 and the sizes of the rational numbers. All sizes are multiplied by a suitable integer to get integer sizes after having the complete list.

The elements of L will belong to four regions. In the first region all the elements have size near $\frac{1}{6}$, in the second one all the sizes are near $\frac{1}{3}$, in the third one each length is $\frac{1}{2}$ and finally in the fourth region all the paths have size 1. In the optimal packing every polyp should contain one element with size 1 and every other arm should contain one element from each other regions. In the list they will be ordered so that all the paths from the first region in a given order are first, then paths from region two and so on. There may also be a constant number of exceptional paths packed to a constant number of polyps.

The exact sizes should be determined so that in the First Fit algorithm $5p+1$ paths with size about $\frac{1}{6}$ fit into the first polyp, five to each arm and one through the center. The rest of the paths with size about $\frac{1}{6}$ will fit into the next few polyps the same way. The paths with sizes near $\frac{1}{3}$ and $\frac{1}{2}$ will fit into the next number of polyps in a similar way, $2p+1$ and $p+1$ to each one, respectively. Finally only one path with size 1 fits to each of the remaining polyps.

Perhaps one may prove that it is possible to determine the exact sizes in such a way. Moreover there may be an algorithm which gives these sizes. Johnson et al. [55] probably used such an algorithm, but since there were only 12 different sizes, it was enough to give the sizes created by the algorithm in their paper.

A somewhat weaker bound will be proved now.

THEOREM 2.7 *If the arm size of the polyp is suitably large then there exists a list L for any large enough M such that $OPT_p(L) \geq M$ and*

$$FF_p(L) \geq OPT_p(L)\left[\frac{p-1}{42p} + \frac{p-1}{6p+1} + \frac{p-1}{2p+1} + \frac{p-1}{p+1} + 1\right].$$

Note that this ratio is still very close to the bound in Theorem 2.1. For some values of p this gives the following lower bounds for the ratio (the upper bounds are given by Theorem 2.1):

$$\begin{aligned} 2.071 &< R_4 (< 2.227) \\ 2.178 &< R_5 (< 2.318) \\ 2.253 &< R_6 (< 2.380) \\ 2.502 &< R_{15} (< 2.570) \\ 2.6904 &< \lim_{i\to\infty} R_i (\leq 2.7) \end{aligned}$$

Unfortunately the asymptotic result still has a 0.01 gap.

PROOF: Let N be a positive integer divisible by $42p, (6p+1), (2p+1)$ and $(p+1)$ and let ε denote $\frac{1}{s+1}$ where s, the length of the arms of the polyps, is chosen suitably large. The list of paths denoted by L consists of the following regions of paths in the order of appearance

a) $N(p-1)$ paths of size $\frac{1}{42} - \varepsilon$,
b) $N(p-1)$ paths of size $\frac{1}{7}$,
c) $N(p-1)$ paths of size $\frac{1}{3}$,
d) $N(p-1)$ paths of size $\frac{1}{2}$,
e) N paths of size 1.

If these paths are packed with the First Fit algorithm then each arm of the first $\frac{N(p-1)}{42p}$ polyp will contain 42 paths of region a). The coarseness of the next polyp will be $(2 \cdot 42 - 1) \cdot \varepsilon$. We choose s large enough, and thus ε small enough, to exclude that one of the later coming paths could fit in. (Note that sum of the sizes of two arms is $2 - \varepsilon$). Namely $83\varepsilon < \frac{1}{7}$ must be true. In this way all the paths in region b) will go to later polyps. Exactly 6 will fit to the first arm, then 6 to the second one etc., and finally 1 through the center. This makes $6p + 1$ in one polyp. Thus region b) will fill up $\frac{N(p-1)}{6p+1}$ polyps. The coarseness of the next polyp will be $\frac{1}{7} - \varepsilon$ so no later paths will fit into these polyps. In the same way region c) will fill up $\frac{N(p-1)}{2p+1}$ polyps and region d) $\frac{N(p-1)}{p+1}$ polyps. Finally only one paths of size 1 may fit into a polyp. Thus we have additional N polyps filled. In this way

$$N\left[\frac{p-1}{42p} + \frac{p-1}{6p+1} + \frac{p-1}{2p+1} + \frac{p-1}{p+1} + 1\right]$$

polyps are needed to pack these paths.

If an optimal packing of list L is considered, it is possible to pack the list into N polyps. Put one path from region e) into the first arm such that it completely covers the first arm and the center. Then put one path from each other region to each arm. In this way one polyp is completely filled. Follow the same procedure until there are some paths left. It is easy to see that N polyps are needed. This packing is optimal since the sum of the sizes of the paths is equal to the sum of the sizes of the polyps. ■

2.2 Applications and Open Problems

We have seen that although it looks very hard to find an optimal packing, the very easily applicable First Fit algorithm works quite well. So if we can model some practical problem by the Polyp Packing one might use the First Fit algorithm.

The vertex disjoint p–Polyp Packing may be considered as a variant of the Bin Packing problem. Now we have groups of bins each group containing p bins. It is possible to break at most one item into two pieces and place the two parts into two different bins of the same group, but it is allowed to do this at most once in each group. In [14], [15], [42], [47], [48] and [55] the reader may find some problems which are modeled by the Bin Packing problem. With this modification we may model some modified problems. For example file allocation. It is desired to place files of varying sizes on as few disks as possible, where files may not be broken between disks but it is allowed to break some files between two tracks of the same disk.

There are some open problems left. First of all our results are not sharp so they may be improved.

There are algorithms for the Bin Packing problem which give packing closer to the optimal. Probably such an algorithm would give a better result here also. Moreover, it is proved in [38] that for every positive ε there exists an $O(n)$-time algorithm for Bin Packing that uses at most $(1 + \varepsilon)$ times more bins than the optimal packing. This may be true for Polyp Packing, too.

Further investigation of this problem may include other graphs not just polyps. However, there should be some restrictions. If we want to pack arbitrary paths into arbitrary graphs then, with any algorithm, when we want to pack a particular path, it must be decided that this path fits into the graph or not. But this problem is $\mathcal{NP}$–complete in general, since it contains the Hamiltonian path problem as a special case. One possible restriction is that the sizes of the paths have a constant upper bound. For example if all paths are at most 100 long, this problem does not arise.

However, there is no such problem in the edge disjoint case, because there is a polynomial algorithm to decide if it is possible to pack a given path to a graph such that it may cross a vertex more than once but it may cover each edge at most once. This problem is equivalent to the Chinese Postman problem which is solved in polynomial time in [32].

Chapter 3

f–Hamiltonian Cycles

3.1 Introduction to f–Hamiltonian Cycles

DEFINITION 3.1 *Let f be a continuous, monotone non-decreasing, non-negative function. Let G be a graph with n vertices. A cycle C of G is called f–Hamiltonian if the distance of every vertex of G from some vertex of C is at most $f(n)$. An f–Hamiltonian path is defined analogously. A graph is said to be f–Hamiltonian if it contains an f–Hamiltonian cycle.*

Deciding whether a graph has an f–Hamiltonian cycle, a problem posed by G. O. H. Katona, is $\mathcal{NP}$–complete for a general f (put $f \equiv 0$). We prove $\mathcal{NP}$–completeness for the case $f(n) \le n^{1-\varepsilon}$ (Theorem 3.2) and polynomial solvability for $f(n) \ge \frac{n}{c}$ (Theorem 3.4). These results hold for f–Hamiltonian paths as well. However, f–Hamiltonian paths always exist if $f(n) \ge \frac{n-4}{3}$ (Theorem 3.6).

As usual, $\lfloor x \rfloor$ denotes the largest integer not larger than x.

3.2 $\mathcal{NP}$–complete Problems

THEOREM 3.2 *The problem is $\mathcal{NP}$–complete if $0 < f(n) \le n^{1-\varepsilon}$ holds for some ε and for all $n \ge n_0$.*

LEMMA 3.3 *If the above function f and the above numbers ε, n_0 are given then, for any positive integer h there exist some non-negative integers s and u satisfying $f(n) \le \frac{n}{\sqrt[k]{n}} = n^{1-\varepsilon}$ for all $n > n_0$ there exist non-negative integers s and u such that $\lfloor f\big((s+1)h+u\big)\rfloor = s, \quad s \le 2^k h^{k-1}$ and $u \le 2^k h^{k-2}$.*

PROOF OF LEMMA 3.3: Consider the function $g(t) = f(t) - \left(\frac{t}{h} - 1\right)$. Clearly, $g(0) > 0$ and $g(t) < 0$ if t is large enough since $f(t) \le \frac{t}{\sqrt[k]{t}} \le \frac{t}{h} - 1$. Let t_0 be the smallest number satisfying $g(t_0) = 0$; such numbers exist by Bolzano's Theorem. Let us investigate the behavior of the functions $f(t)$ and $\frac{t}{h} - 1$ in the interval $[\lfloor t_0 \rfloor, t_0]$. Observe first that $\lfloor \frac{t_0}{h} - 1 \rfloor = \left\lfloor \frac{\lfloor t_0 \rfloor}{h} - 1 \right\rfloor$, for otherwise $\frac{t}{h} - 1$ should by its monotonicity be integer at an internal point of the interval.

Applying the monotonicity of $f(t)$ and the relation $\frac{t}{h} - 1 \le f(t)$ in this interval we obtain $\left\lfloor \frac{\lfloor t_0 \rfloor}{h} - 1 \right\rfloor \le \lfloor f(\lfloor t_0 \rfloor) \rfloor \le \lfloor f(t_0) \rfloor = \lfloor \frac{t_0}{h} - 1 \rfloor$.

Now define s and u by $\lfloor t_0 \rfloor = (s+1)h + u \quad (0 \le u < h)$. Then $\lfloor f\big((s+1)h+u\big) \rfloor = s$ clearly follows. In order to see the remaining properties, suppose indirectly that $t_0 > 2^k h^k, n_0$. Then

$$\frac{t_0}{\sqrt[k]{t_0}} \ge f(t_0) = \frac{t_0}{h} - 1 > \frac{t_0}{\frac{1}{2}\sqrt[k]{t_0}} - 1$$

implies

$$\frac{t_0}{\sqrt[k]{t_0}} \le 1$$

a contradiction. ■

Proof of Theorem 3.2: We reduce the problem "Decide whether a given graph H is Hamiltonian" to our present one. Let H contain h vertices; choose s and u as in Lemma 3.3. Attach a path of length s each to every vertex of H and additional u paths of length one to one distinguished vertex (see Fig. 3.1).

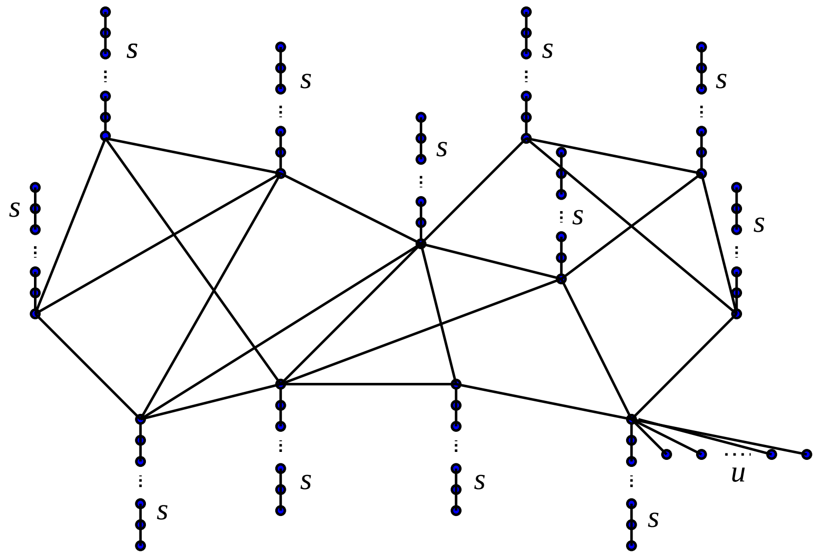

Fig. 3.1

The new graph G has $(s+1)h + u$ vertices (which is a polynomial function of h). We claim that G is f–Hamiltonian if and only if H was Hamiltonian. The "if" part is obvious since $\lfloor f\big((s+1)h+u\big) \rfloor = s$. For the "only if" part all we have to observe is that an f–Hamiltonian cycle C of G is a cycle, hence does not contain any of the new edges and if C did not contain a vertex x of H then the other end of the path attached to x would violate f–Hamiltonicity. ■

3.3 Polynomially Solvable Problems

THEOREM 3.4 *The problem is polynomially solvable if $f(n) \geq \frac{n}{c}$ holds for some positive integer c and for all $n > n_0$.*

LEMMA 3.5 *The graph G has an f–Hamiltonian cycle C if and only if there is a set R of at most $c+1$ vertices, contained in a cycle C' so that the distance of every vertex of G from some vertex of R is at most $f(n)$.*

PROOF OF LEMMA 3.5: The "if" part is obvious since C' can play the role of C. Conversely, we construct R from C. Let $c_1, c_2, \ldots, c_t$ be the vertices of C in this order. For every vertex v of G, determine the closest vertex c_i of C. Denote by C_i the set of those vertices of G which are closest to c_i. The relation $c_i \in C_i$ is obvious. After this we build the breadth first spanning tree of the subgraph induced by C_i from c_i, as the root of the tree. First we connect c_i with the vertices which are adjacent to c_i in G. Then we connect one of these vertex with the vertices adjacent to it in G, but not connected earlier to any other vertex. And so on, until we have connected every vertex to the tree. Since the graph is connected we connect every point to some other one. Obviously we obtain a tree F_i for each i. Let F denote the union of the cycle C and the trees F_i (see Fig. 3.2).

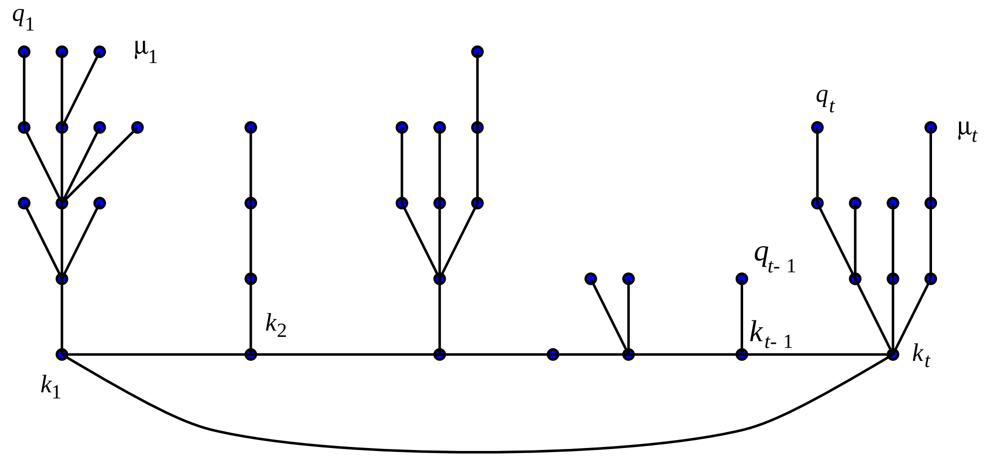

Fig. 3.2

These trees determine the shortest paths between the cycle and the vertices of the graph. Thus it is easy to determine which vertex of F_i is the farthermost from the root c_i. Denote this vertex by q_i and let μ_i its distance from c_i. Define

$$\omega_1 = \max_{1 \leq j \leq t} \{j | \quad \mu_i + j - i \leq f(n) \quad \text{for every} \quad 1 \leq i \leq j\}.$$

This means that c_{ω_1} has the property, that if $i \leq \omega_1$ then none of the vertices of F_i is farther from c_{ω_1} than $f(n)$ in the graph G, since there is such a path between c_{ω_1} and the vertices of F_i. On the other hand there exists a vertex among the vertices of F_i for some $i \leq \omega_1$ such that its distance from c_{ω_1} is exactly $f(n)$ in F. Now delete the vertices of the trees F_i from the graph F for all $i \le \omega_1$. For the remaining graph, determine ω_2 exactly like ω_1 before. Repeating this step, we obtain the subscripts $\omega_3, \ldots, \omega_m$. All the statements for c_{ω_1} are true for every c_{ω_i}, too, except for c_{ω_m}, for which it is not sure that there exists a vertex with distance exactly $f(n)$ from it. Thus $R = \{c_{\omega_1}, c_{\omega_2}, \ldots, c_{\omega_m}\}$ satisfies the conditions that every vertex of G can be reached from one of the vertices of the set R on a path not longer than $f(n)$, and there is a cycle in G, namely C, which contains all the vertices of the set R.

Thus there exist vertex disjoint paths of length exactly $f(n)$ starting from $c_{\omega_1}, c_{\omega_2}, \ldots, c_{\omega_{m-1}}$, resp. So the number of the vertices in the graph G is at least $(m-1)f(n)$. Hence we have $(m-1)f(n) \le n$. The condition $f(n) \ge \frac{n}{c}$ implies $m \le c+1$. ■

Proof of Theorem 3.3: Let us check whether the condition of Lemma 3.5 holds. Recall that our upper bound for the number of steps must be a polynomial in the number n of the vertices but the coefficients of this polynomial may contain c in the exponent. If we choose a set of $d \le c+1$ vertices arbitrarily we can easily check whether every vertex can be reached from these ones along a path of length at most $f(n)$. All we have to check is whether there is a cycle containing all the vertices of R.

Robertson and Seymour [77] proved that if $p_1, p_2, \ldots, p_d, q_1, q_2, \ldots, q_d$ are given vertices of a graph then one can decide in polynomial time whether there are d vertex disjoint paths connecting p_1 to q_1, p_2 to q_2 and so on. Apply this algorithm with $p_1 = v_1, q_1 = p_2 = v_2, q_2 = p_3 = v_3, \ldots, q_d = v_1$ where $v_1, v_2, \ldots, v_d$ are the elements of R (in some order). Since we can choose R in any way and try its elements in any order, our algorithm is polynomial. ■

These Theorems are true for f–Hamiltonian paths as well, with slight modifications in the proofs. The following additional result may be of some interest.

Theorem 3.6 *If $f(n) > \frac{n-4}{3}$, then there is an f–Hamiltonian path in the graph.*

Proof: Let $t_1, t_2, \ldots, t_d$ be the longest simple path in the graph. We prove that this is an f–Hamiltonian path. Assume indirectly that there is a vertex p which is not reachable from this path along a path not longer than $f(n)$. Let t_i be the closest vertex to p from the vertices of the path. This means that the length l of the shortest path from t_i to p satisfies $l > f(n)$. Suppose that $i-1 < l$. Then the simple path from p to t_d via t_i is longer than $(t_1, \ldots, t_d)$, a contradiction. We obtained $i-1 \ge l$. The inequality $d-i \ge l$ is obtained in the same way. The total number of vertices along these paths is $d+l \ge (i-1)+(d-i)+1+l \ge 3l+1 \ge 3(f(n)+1)+1$. This contradicts our assumption $\frac{n-4}{3} < f(n)$. ■

The two examples of Fig. 3.3 with $n = 10$, $f(10) = 2$ show that the algorithm is really needed if $f(n) \le \frac{n-4}{3}$.

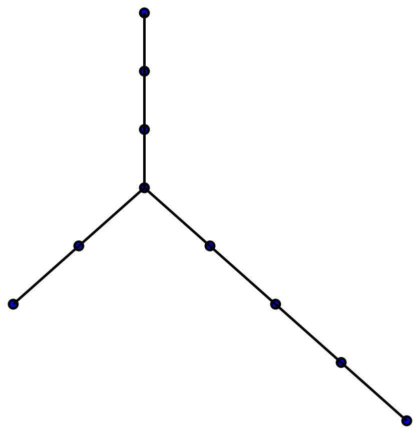

Fig. 3.3/a

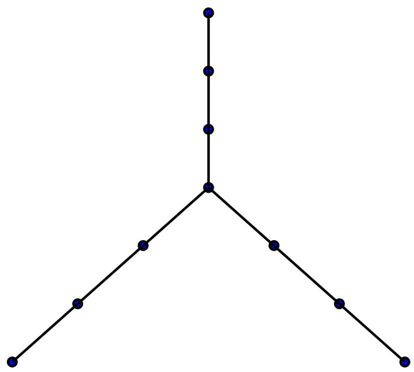

Fig. 3.3/b

Chapter 4

Toughness and Edge–toughness

4.1 Introduction to Edge–toughness

In 1973 Chvátal [18] introduced the definition of toughness. Many authors investigated the relation of toughness and Hamiltonicity since Chvátal's paper but his famous Conjecture is still open. A good survey of the results from this area is [9]. The notations used in this Chapter are given next.

Let $\omega(G)$ denote the number of components of a graph G. If $T \subseteq V(G)$ then the graph $G - T$ is defined as follows. $V(G-T) = V(G) - T$ and $(u, v) = e \in E(G-T)$ iff $e \in E(G)$ but non of u and v is in T. A graph G is *t–tough* if $|S| \geq t\omega(G-S)$ for every subset S of the vertex set $V(G)$ with $\omega(G-S) > 1$. The *toughness* of G, denoted by $t(G)$, is the maximum value of t for which G is t-tough (taking $t(K_n) = \infty$ for all $n \geq 1$). A *k–factor* of a graph is a k–regular spanning subgraph. It is worth mentioning that a t–tough graph is always $2t$-vertex-connected.

Chvátal investigated the relation of toughness and Hamiltonicity in [18]. Being 1–tough is clearly a necessary condition for a graph to be Hamiltonian. He conjectured that there exists a finite constant t_0 such that every t_0–tough graph is Hamiltonian. This Conjecture is still open. Enomoto, Jackson, Katerinis and Saito in [34] showed that if there is such a t_0 then $t_0 \geq 2$. In fact, it is a consequence of a more general result which was proved in their paper.

THEOREM 4.1 [Enomoto et al.] *Let $k \geq 1$. For any $\varepsilon > 0$ there exists a $(k - \varepsilon)$–tough graph G on v vertices with $v \geq k + 1$ and kv even which has no k–factor.*

For $k = 2$ this means that $(2 - \varepsilon)$–tough graphs were shown which have no 2-factor and therefore they are not Hamiltonian. Hence if there exists a t_0 then $t_0 \geq 2$.

On the other hand they also proved in [34] that this result is sharp.

THEOREM 4.2 [Enomoto et al.] *Let G be a k–tough graph on v vertices with $v \geq k+1$ and kv even. Then G has a k–factor.*

If we are trying to show that $t_0 > 2$, then we must show that there is a 2–tough graph which is not Hamiltonian. How can we prove that the graph is not Hamiltonian? The result of Enomoto et al. shows that proving that there is no 2–factor will not work. Therefore we have to find some other method to prove non–Hamiltonicity.

Another method would be showing non–path–toughness [28]. Let $l(G)$ be the minimum number of vertex disjoint paths that are needed to cover all the vertices of G and $r(G) = \max\{l(G-X) - |X| | X \subseteq V(G) \text{ and } l(G-X) \neq 1\}$. A graph G is *path–tough* iff $r(G) \leq 0$. But unfortunately there is no general method for showing non–path–toughness.

Mader in [73] and later Lomonosov in [72] investigated the question of cycles through prescribed elements of a graph. A special case of this problem is of course the Hamiltonian cycle problem where the set of prescribed elements of the graph is the set of all vertices. Some definitions and a Lemma will be presented here from this paper. The general form is not needed, therefore only a special case is given here.

In general the A–cycle problem is to decide whether there exists a cycle containing all elements of A, where A is a collection of vertices and edges from the graph. But let us restrict ourselves to the case when $A \subseteq V(G)$. For sake of simplicity we will suppose that the elements of A are pairwise non-adjacent. Otherwise a 2-degree vertex is put into the middle of the edge connecting the two vertices. (This new vertex is not in A in this case.)

We say that a pair (X, Y), where $X \subseteq V(G - A)$ and $Y \subseteq E(G - A - X)$, *disconnects* A, if $G - X - Y$ has no path connecting two vertices of A. ($G - X - Y$ is obtained by deleting the vertices of X and the edges of Y from G, but the endvertices of the edges in Y are in the graph $G - X - Y$.) Let $Q_1, Q_2, \ldots, Q_n$ be the components of the edge–induced subgraph $G(Y)$. The vertices of $G(Y)$ are the vertices of G which are incident to an edge in Y and the edges of $G(Y)$ are those edges of G whose endvertices are both in the vertex set of $G(Y)$. Therefore there may be edges in $G(Y)$ which are not in Y. Let H be a graph and B a subgraph of H, then the boundary of B in H is defined by $\mathrm{bd}_H(B) = \{v \in V(B) | v \text{ has a neighbour outside of } B\}$ and $\mathrm{in}_H(B) = V(B) - \mathrm{bd}_H(G)$. Therefore in our case

$$\mathrm{bd}_{G-X}(Q_i) = \{v \in V(Q_i) \mid v \text{ has a neighbour outside of } Q_i \text{ and } X\}$$

and

$$\mathrm{in}_{G-X}(Q_i) = V(Q_i) - \mathrm{bd}_{G-X}(Q_i).$$

The following Lemma may be formulated.

LEMMA 4.3 (see [73]) *If there exists a pair (X, Y) disconnecting A, such that*

$$|A| > s(X, Y; G) := |X| + \sum_{i=1}^{n} \left\lfloor \frac{|\mathrm{bd}_{G-X}(Q_i)|}{2} \right\rfloor, \tag{4.1}$$

(where $\lfloor x \rfloor$ denotes the largest integer satisfying $\leq x$,) then there is no cycle containing all vertices of A.

A pair (X, Y) which satisfies (4.1) is called an A–*separator*. Fig. 4.1 presents some examples of A–separators.

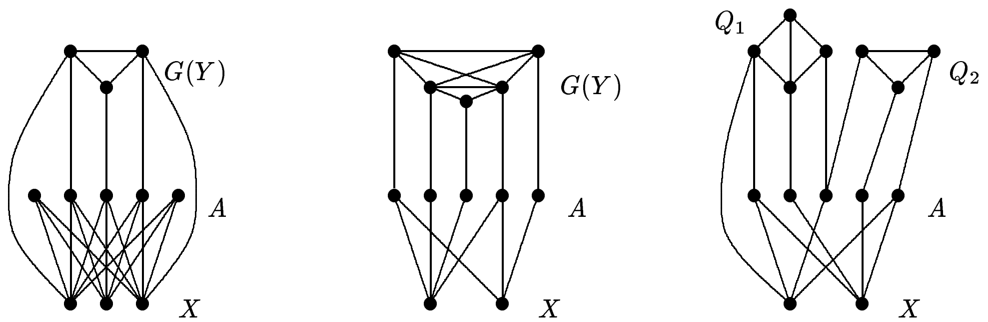

Fig. 4.1

Lomonosov investigates A-separators in [72], our aim is different, we are interested in Hamiltonian cycles.

Let us call a path in a graph G an AA–path iff its endvertices are different vertices in A, and let $m(A,G)$ denote the number of all disjoint AA–paths in G. For a pair (X,Y) being an A–separator in G, each AA–path in G must contain a vertex from X or an edge from Y, in the latter case therefore two vertices of some Q_i. But only $\left\lfloor \frac{|\mathrm{bd}_{G-X}(Q_i)|}{2} \right\rfloor$ disjoint AA–paths can cross the component Q_i, thus

$$m(A,G) \leq s(X,Y;G).$$

Considering (4.1) this implies that there is no cycle in G containing all vertices of A, which proves Lemma 4.3.

If (X,Y) is an A–separator in G then obviously, $(\emptyset,Y)$ is an A–separator in the graph $G-X$ with $s(\emptyset,Y;G-X) = s(X,Y;G) - |X|$, and we get analogously

$$m(A,G-X) \leq s(\emptyset,Y;G-X) = s(X,Y;G) - |X|. \tag{4.2}$$

This implies that $l(G-X) > |X|$, since if one could cover $G-X$ (including all vertices of A) with $r \leq |X|$ vertex disjoint paths then there must be at least $|A| - r \geq |A| - |X|$ disjoint AA–paths in $G-X$, i. e. $m(A,G-X) + |X| \geq |A|$, in contradiction to (4.2) and (4.1). This proves the following Lemma.

LEMMA 4.4 *If there exists an A-separator in G then G is not path–tough.*

Note that presenting an A-separator is an easy way of proving non–path–toughness in a graph.

The following Theorem from [31] will be used in the proof of Theorem 4.15.

THEOREM 4.5 [Edmonds] *G has a 2–factor iff*

$$|V'| \geq \frac{1}{2}\left(|K| + \kappa\right)$$

holds for all $V' \subseteq V(G)$. K denotes the set of vertices having degree at most one in $G-V'$ and κ denotes the number of components of $G-V'$ containing an odd number of vertices in K.

4.2 t–edge–toughness

Obviously, if there is a subset of vertices A such that there exists an A–separator then there is certainly no Hamiltonian cycle in the graph. This is a new method to prove non–Hamiltonicity. It is a generalization of the previous method where non-Hamiltonicity is proved with proving non-1-toughness. The "proof" for non-Hamiltonicity is the cut set in that case, while the "proof" in our case is a "cut set" consisting of vertices and edges. However, the set A is not really needed for this, therefore we give a new definition here.

DEFINITION 4.6 *G is t–edge–tough if G is connected and*

$$\omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) \leq \frac{s(X,Y;G)}{t}$$

holds for every $X \subseteq V(G)$ and $Y \subseteq E(G-X)$ satisfying $\omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) > 1$, using the notations introduced above. The edge–toughness of G, denoted by $t_e(G)$, is the maximum value of t_e for which G is t_e–edge–tough (taking $t_e(K_n) = \infty$ for all $n \geq 1$).

Two observations are immediate.

OBSERVATION 4.7 *If G is t–edge–tough then it is t–tough.*

PROOF: The assertion follows from the definition above taking $Y = \emptyset$. ■

OBSERVATION 4.8 *If G is Hamiltonian then it is 1–edge-tough.*

PROOF: It is enough to show that if G is not 1–edge–tough then there is a vertex set A and a pair (X, Y), such that this pair is an A–separator. Obviously if G is not 1–edge–tough then there exists a pair (X, Y), such that

$$\omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) > s(X, Y; G).$$

It will be proved that every component of $G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)$ contains at least one vertex which is disjoint from each $\mathrm{bd}_{G-X}(Q_i)$. Suppose that there is a component which contradicts our claim. All vertices in this component are in one of the vertex sets $\mathrm{bd}_{G-X}(Q_i)$. It is easy to see that there is no edge connecting a vertex of Q_i to a vertex of Q_j where $i \neq j$, since Q_i and Q_j are different components in $G(Y)$. If v is a vertex of our component and $v \in \mathrm{bd}_{G-X}(Q_p)$ then v must have a neighbor outside of Q_p and X by definition and, as we have seen, this neighbor cannot be in $\mathrm{bd}_{G-X}(Q_j)$ for any j, either. Hence this neighbor is disjoint from each $\mathrm{bd}_{G-X}(Q_i)$, a contradiction.

To construct A chose one vertex from each component of $G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)$ such that each of them is disjoint from each $\mathrm{bd}_{G-X}(Q_i)$. All edges in Y are edges of some Q_i therefore the vertices of A are not incident to any edge in Y. This implies that the elements of A are pairwise nonadjacent. On the other hand it is easy to see that (X, Y) is an A-separator. ■

4.3 Connection Between Toughness and Edge–toughness

It may be possible to construct a non–1–edge–tough, but 2–tough graph using Observation 4.8. Such a graph would show that if there exists a t_0 in Chvátal's Conjecture then $t_0 > 2$.

But this is not the case, the following Theorem can be proved, instead.

THEOREM 4.9 *If G is $2t$–tough then it is t–edge–tough.*

PROOF: For any $X \subseteq V(G)$ and any $Y \subseteq E(G - X)$ satisfying

$$\omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) > 1,$$

let S consist of all vertices of X and for every $i = 1, \ldots, n$ of $2\left\lfloor \frac{|\mathrm{bd}_{G-X}(Q_i)|}{2} \right\rfloor$ vertices of $\mathrm{bd}_{G-X}(Q_i)$. Obviously

$$|S| = |X| + 2\sum_{i=1}^{n} \left\lfloor \frac{|\mathrm{bd}_{G-X}(Q_i)|}{2} \right\rfloor = 2s(X, Y; G) - |X|.$$

It will be shown now that

$$\omega(G - S) \geq \omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right)$$

holds. A set A is constructed containing exactly one vertex from each component of $G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)$ and containing no vertex from any $\mathrm{bd}_{G-X}(Q_i)$. It was shown in the proof of Observation 4.8 that every component contains such a vertex. It is easy to see that A satisfies $A \cap S = \emptyset$. In this way all vertices of A are vertices of $G - S$. It will be shown that the vertices of A are contained in different components of $G - S$, which proves our claim. Suppose there is a path in $G - S$ connecting two vertices of A. Since (X, Y) disconnects A this path must contain a vertex from X or an edge from Y. $X \subseteq S$, therefore the path cannot contain a vertex from X. Suppose now that the path contains an edge from Y and that this edge is an edge of Q_p. The first and the last edge of the path cannot be in Y, and therefore in Q_p, since A is not incident to any edges from Y by our construction. This implies that there must be at least two inner vertices of this path each of which is incident to an edge from Q_p and to an edge which is not in Q_p. This implies that these two vertices must be elements of $\mathrm{bd}_{G-X}(Q_p)$. By our construction of S, $G - S$ contains at most one of these two vertices, therefore there is no path connecting two vertices of A.

Now because G is $2t$–tough we get

$$\omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) \leq \omega(G - S) \leq \frac{|S|}{2t} = \frac{2s(X, Y; G) - |X|}{2t} \leq \frac{s(X, Y; G)}{t},$$

thus G is t–edge-tough. ■

We conjecture that the converse of Theorem 4.9 is true for no t.

Conjecture 4.10 *For every $t > 0$ and $\varepsilon > 0$ there exists a graph G such that G is t–edge–tough and $t(G) < t + \varepsilon$.*

For $t = 1$ such graph is a complete split graph $S_{n+1,n}$ defined as follows.

Let $\{a_1, \ldots, a_{n+1}, b_1, \ldots b_n\}$ be the set of vertices. Let $\{a_1, \ldots, a_{n+1}\}$ span a complete graph K_{n+1} and $\{b_1, \ldots, b_n\}$ a completely nonconnected graph $\overline{K_n}$. Connect a_i and b_j for all i, j. This graph is 1–edge–tough, because it contains a Hamiltonian cycle (Observation 4.8). On the other hand $t(S_{n+1,n}) = \frac{n+1}{n}$.

It is proved next that Theorem 4.9 is sharp.

Theorem 4.11 *For any positive integer t and any positive real number $\varepsilon < 1$ there exist $(2t - \varepsilon)$–tough but non–t–edge–tough graphs.*

Proof: Let p be a positive integer such that $p \geq \frac{t}{\varepsilon}$. The construction of the graph $G_{t,p}$ satisfying the conditions of the Theorem is given below.

The vertex set $V(G_{t,p})$ of the graph is the union of the following sets (see Fig. 4.2).

$$\begin{aligned}
X &= \{x_1, x_2, \ldots, x_{2t}\}, \\
A &= \{a_1, a_2, \ldots, a_{(2t+3)(2p+1)}\}, \\
Q_i &= \{q_{(2p+1)(i-1)+1}, q_{(2p+1)(i-1)+2}, \ldots, q_{(2p+1)(i-1)+2p+1}\} \text{ for } 1 \leq i \leq (2t+3), \\
R_i &= \{r_{i,1}, r_{i,2}, \ldots, r_{i,(2t+3)(2p+1)}\} \text{ for } 1 \leq i \leq (2t-1).
\end{aligned}$$

For shorter formulas let $\alpha = (2t+3)(2p+1) = 4tp + 6p + 2t + 3$, $\mathcal{Q} = \cup_1^{2t+3} Q_i$ and $\mathcal{R} = \cup_1^{2t-1} R_i$. Thus $V(G_{t,p}) = X \cup A \cup \mathcal{Q} \cup \mathcal{R}$.

The edges of $G_{t,p}$ are given next. All vertices of X are adjacent to all other vertices of $G_{t,p}$ including all other vertices of X. For each $1 \leq i \leq (2t+3)$ Q_i spans a complete graph K_{2p+1}. For each $1 \leq i \leq (2t-1)$ R_i spans a complete graph K_α. The neighborhood of a_i is defined

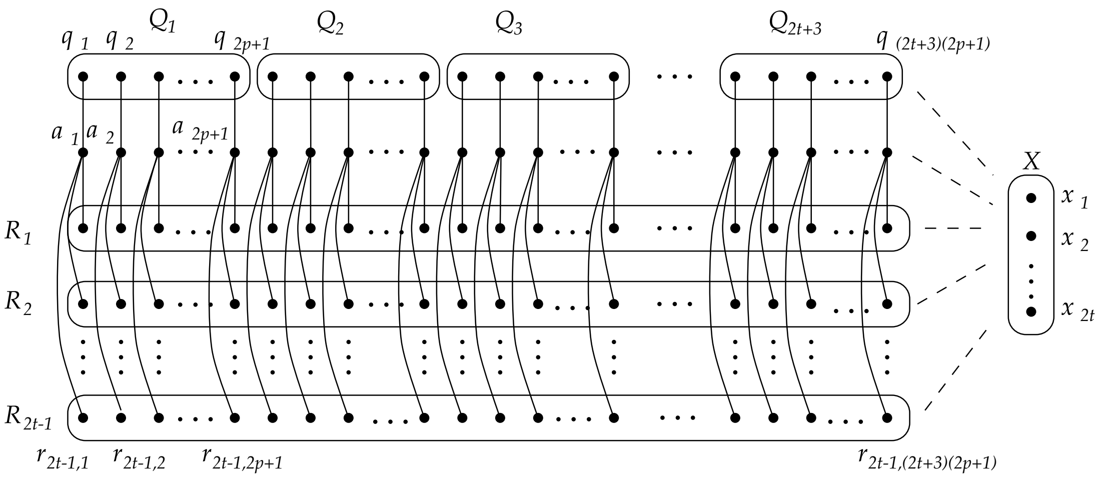

Fig. 4.2

by $N(a_i) = \{q_i, r_{1,i}, r_{2,i}, \ldots, r_{2t-1,i}\} \cup X$. Note that A spans an empty graph and that there are no edges between R_i and R_j if $i \neq j$ and between $\mathcal{Q}$ and $\mathcal{R}$.

First we prove that $G_{t,p}$ is not t–edge–tough. Since it does not cause misunderstandings, G is used instead of $G_{t,p}$ from here. Let Y be the set of edges spanned by $\mathcal{Q}$ and $\mathcal{R}$. Thus $G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)$ consists of disjoint stars spanned by $\{a_i, q_i, r_{1,i}, \ldots, r_{2t+1,i}\}$ where $1 \leq i \leq \alpha$.

Therefore $\omega\,(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)) = \alpha$. The components of the edge-induced subgraph $G(Y)$ are $Q_1, Q_2, \ldots, Q_{2t+3}, R_1, R_2, \ldots, R_{2t-1}$. $\mathrm{bd}_{G-X}(Q_i) = Q_i$ and $\mathrm{bd}_{G-X}(R_i) = R_i$ thus $|\mathrm{bd}_{G-X}(Q_i)| = 2p+1$, $|\mathrm{bd}_{G-X}(R_i)| = \alpha$ and $\mathrm{in}_{G-X}(Q_i) = \mathrm{in}_{G-X}(R_i) = \emptyset$.

It must be proved that

$$\omega\,(G - X - Y) = \alpha > \frac{2t + (2t+3)\left\lfloor \frac{2p+1}{2} \right\rfloor + (2t-1)\left\lfloor \frac{\alpha}{2} \right\rfloor}{t} = \frac{s(X, Y; G)}{t}.$$

This is equivalent to

$$t(2t+3)(2p+1) > 2t + (2t+3)p + (2t-1)(2pt + 3p + t + 1)$$

which is clearly true.

To complete the proof we have to prove that G is $(2t - \varepsilon)$–tough. Suppose indirectly that there exists a subset of vertices S such that $(2t - \varepsilon)\omega(G - S) > |S|$ and $\omega(G - S) > 1$. The latter condition implies $X \subseteq S$.

Let $\mathcal{C}_{\mathrm{a}}$ be the set of those components of $G - S$ which contain vertices only from A. This implies that these components are isolated vertices since there are no two connected vertices in A. Let $\mathcal{C}_{\mathrm{b}}$ denote the set of the rest of the components of $G - S$. Each of these components must contain a vertex from $V(G) - A$. Finally denote $|\mathcal{C}_{\mathrm{a}}|$ and $|\mathcal{C}_{\mathrm{b}}|$ by c_{a} and c_{b}, resp. Thus $c_{\mathrm{a}} + c_{\mathrm{b}} = \omega(G - S)$.

Case 1 Suppose that $0 \leq c_{\mathrm{b}} \leq 1$. If a_i is an isolated vertex in $G - S$ then $N(a_i) \subseteq S$. Since $N(a_i) \cap N(a_j) = \emptyset$ if $i \neq j$ we obtain $|S| \geq |X| + 2tc_{\mathrm{a}}$ because $|N(a_i)| = 2t$. Therefore

$$|S| \geq 2t + 2tc_{\mathrm{a}} \geq 2t(1 + c_{\mathrm{a}}) \geq 2t\omega(G - S) > (2t - \varepsilon)\omega(G - S),$$

a contradiction.

Case 2 This case covers all the other cases, namely when $c_{\mathrm{b}} \geq 2$ and $c_{\mathrm{a}} \geq 0$. There are 3 subcases:
Case 2.1 There exist 2 different components of $G-S$ which contain at least one vertex of $\mathcal{R}$.
Case 2.2 Exactly one component of $G-S$ contains vertices of $\mathcal{R}$.
Case 2.3 No component of $G-S$ contains any vertex of $\mathcal{R}$.

Case 2.1 Two vertices of the same Q_i or R_i cannot be in different components of $G-S$ thus

$$\omega(G-S) \leq c_{\mathrm{a}} + (2t+3) + (2t-1) = c_{\mathrm{a}} + 4t + 2.$$

As in Case 1, S must contain $N(a_i)$ if a_i is a component in $G-S$. Let S_1 be the union of these neighborhoods. This implies $S_1 \subseteq S$. It is easy to see that $|S_1| \geq 2tc_{\mathrm{a}}$.

Let $r_{n,m}, r_{k,l} \in V(G-S)$ where $n \neq k$ and such that they are in different components of $G-S$. The paths $(r_{n,m}, r_{n,i}, a_i, r_{k,i}, r_{k,l})$ for $1 \leq i \leq \alpha$ are vertex disjoint paths each connecting $r_{n,m}$ and $r_{k,l}$. (If two consecutive vertices of this path coincide then replace it with one of them.) Since these two vertices are in different components of $G-S$, S must contain at least one vertex from each of these paths. Let S_2 denote a set of vertices that contains exactly one vertex from each path, thus $|S_2| = \alpha$. To have a lower bound on $|S|$ we need an upper bound on $|S_1 \cap S_2|$. By the construction of the graph, $N(a_i)$ and $(r_{n,m}, r_{n,i}, a_i, r_{k,i}, r_{k,l})$ have two common vertices but S_2 may contain at most one of them. This implies that each different neighborhood may have at most one vertex in S_2 which gives the conclusion $|S_1 \cap S_2| \leq \frac{|S_1|}{2t}$.

Since $X \subset S$, we obtain

$$|S| \geq |X| + |S_1| + |S_2| - |S_1 \cap S_2| \geq |X| + \left(1 - \frac{1}{2t}\right)|S_1| + |S_2| \geq 2t + (2t-1)\,c_{\mathrm{a}} + (2t+3)(2p+1).$$

Therefore, by our indirect assumption

$$(2t-\varepsilon)\,(c_{\mathrm{a}} + 4t + 2) > 2t + (2t-1)\,c_{\mathrm{a}} + (2t+3)(2p+1).$$

which is equivalent to

$$(1-\varepsilon)\,c_{\mathrm{a}} + (2t-\varepsilon)\,(4t+2) - 2t > (2t+3)(2p+1).$$

Since $c_{\mathrm{a}} \leq \alpha = (2t+3)(2p+1)$ we obtain

$$(1-\varepsilon)\,(2t+3)(2p+1) + (2t-\varepsilon)\,(4t+2) - 2t > (2t+3)(2p+1)$$

because $1-\varepsilon \geq 0$. This implies

$$\frac{(2t-\varepsilon)\,(4t+2) - 2t}{2\varepsilon(2t+3)} - \frac{1}{2} > p.$$

It is easy to see that the left hand side is less than $\frac{t}{\varepsilon}$, contradicting our assumption on p.

Case 2.2 Without loss of generality we may suppose that $G - S$ contains $r_{1,1}$. Therefore all the other $c_{\mathrm{b}} - 1$ components of $\mathcal{C}_{\mathrm{b}}$ must contain a vertex from $\mathcal{Q}$. Let $q_{(2p+1)k+l}$ be such a vertex. The paths $(r_{1,1}, r_{1,(2p+1)k+i}, a_{(2p+1)k+i}, q_{(2p+1)k+i}, q_{(2p+1)k+l})$ for $1 \le i \le 2p+1$ are vertex disjoint paths (apart from the endvertices), each connecting $r_{1,1}$ and $q_{(2p+1)k+l}$. (If two consecutive vertices of this path coincide then replace it with one of them.) Therefore S must contain at least one vertex from each of these $2p+1$ paths. This is true for all other components which contain a vertex from $\mathcal{Q}$ and the paths obtained this way are vertex disjoint from each other (apart from the endvertices). Therefore there are $(c_{\mathrm{b}} - 1)(2p+1)$ vertex disjoint paths, each connecting $r_{1,1}$ and a vertex of $\mathcal{Q}$. Thus S must contain at least one vertex from these paths.

Let S_3 denote a subset of S which contains exactly one vertex from each of these paths. It is easy to see that such S_3 exists. Thus $|S_3| = (c_{\mathrm{b}} - 1)(2p+1)$.

As in Case 1, S must contain $N(a_i)$ if a_i is a component in $G - S$. Let S_4 be the union of these neighborhoods. This implies $S_4 \subseteq S$. It is easy to see that $|S_4| = 2tc_{\mathrm{a}}$.

To continue the proof we need a lower bound on $|S_3 \cup S_4|$. By the construction of the graph, $N(a_{(2p+1)k+i})$ and $(r_{1,1}, r_{1,(2p+1)k+i}, a_{(2p+1)k+i}, q_{(2p+1)k+i}, q_{(2p+1)k+l})$ have two common vertices but S_3 may contain at most one of them. This implies that each different neighborhood may have at most one vertex in S_3 which gives the conclusion $|S_3 \cap S_4| \le \frac{|S_4|}{2t}$. Thus $|S_3 \cup S_4| \ge |S_3| + \left(1 - \frac{1}{2t}\right)|S_4|$ holds.

On the other hand $q_{(2p+1)k+l} \in V(G - S)$ thus $N(a_{(2p+1)k+l}) \not\subseteq S$ which implies that $r_{1,(2p+1)k+l} \in S_3 - S_4$ because $(r_{1,1}, r_{1,(2p+1)k+l}, q_{(2p+1)k+l})$ is a path connecting $r_{1,1}$ and $q_{(2p+1)k+l}$. Since the same argument works for all the other components of $\mathcal{C}_{\mathrm{b}}$ except the one which contains $r_{1,1}$ this implies $|S_3 - S_4| \ge c_{\mathrm{b}} - 1$. Therefore $|S_3 \cup S_4| \ge |S_4| + c_{\mathrm{b}} - 1$ also holds. Thus

$$\begin{aligned} |S_3 \cup S_4| \ge \max\left(|S_3| + \left(1 - \frac{1}{2t}\right)|S_4|, |S_4| + c_{\mathrm{b}} - 1\right) = \\ = |S_4| + \max\left(|S_3| - \frac{1}{2t}|S_4|, c_{\mathrm{b}} - 1\right) \end{aligned}$$

is obtained.

Two cases are distinguished:

Case 2.2.1 $|S_3| - \frac{1}{2t}|S_4| \ge c_{\mathrm{b}} - 1$

Case 2.2.2 $|S_3| - \frac{1}{2t}|S_4| < c_{\mathrm{b}} - 1$.

Case 2.2.1 In this case by our indirect assumption we obtain

$$(2t - \varepsilon)\,\omega(G - S) = (2t - \varepsilon)\,(c_{\mathrm{a}} + c_{\mathrm{b}}) > |X| + |S_3 \cup S_4| \ge 2t + 2tc_{\mathrm{a}} + (c_{\mathrm{b}} - 1)(2p+1) - c_{\mathrm{a}}. \quad (4.3)$$

Let $m = |S_3| - \frac{1}{2t}|S_4| = (c_{\mathrm{b}} - 1)(2p+1) - c_{\mathrm{a}}$. In this case $m \ge c_{\mathrm{b}} - 1$ thus $m > 0$ holds. c_{b} can be expressed as a function of m

$$c_{\mathrm{b}} = \frac{m + c_{\mathrm{a}}}{2p+1} + 1.$$

Substituting this into (4.3) we obtain

$$(2t - \varepsilon)\left(c_{\mathrm{a}} + \frac{m + c_{\mathrm{a}}}{2p+1} + 1\right) > 2t + 2tc_{\mathrm{a}} + m$$

or equivalently

$$(2t-\varepsilon)\, c_{\mathrm{a}}\, (2p+2) + (2t-\varepsilon)\, m + (2t-\varepsilon)\, (2p+1) > 2t(2p+1) + 2tc_{\mathrm{a}}(2p+1) + m(2p+1) \quad (4.4)$$

is obtained.

Our assumption on p ensures that $(2t-\varepsilon)\, m < m(2p+1)$. It easy to see that $(2t-\varepsilon)\,(2p+1) < 2t(2p+1)$. With these two observations we obtain

$$(2t-\varepsilon)\, c_{\mathrm{a}}\, (2p+2) > 2tc_{\mathrm{a}}(2p+1)$$

from (4.4). This is a contradiction if $c_{\mathrm{a}} = 0$. If $c_{\mathrm{a}} \geq 1$ then this implies $\frac{t}{\varepsilon} > \frac{t-\varepsilon}{\varepsilon} > p$, contradicting our assumption on p.

Case 2.2.2 In this case by our indirect assumption we obtain

$$(2t-\varepsilon)\, \omega(G-S) = (2t-\varepsilon)\,(c_{\mathrm{a}} + c_{\mathrm{b}}) > |X| + |S_3 \cup S_4| \geq 2t + 2tc_{\mathrm{a}} + c_{\mathrm{b}} - 1$$

or equivalently

$$c_{\mathrm{b}}(2t-1-\varepsilon) - 2t + 1 > c_{\mathrm{a}}\varepsilon. \quad (4.5)$$

The assumption of this case implies $(c_{\mathrm{b}}-1)2p < c_{\mathrm{a}}$ thus from (4.5) we obtain

$$c_{\mathrm{b}}(2t-1-\varepsilon) - 2t + 1 > (c_{\mathrm{b}}-1)2p\varepsilon$$

which implies

$$t > \frac{1}{2}(2t-1-\varepsilon) - \frac{\varepsilon}{2c_{\mathrm{b}}-2} > p\varepsilon,$$

contradicting our assumption on p.

Case 2.3 In this case there may be some components from $\mathcal{C}_{\mathrm{a}}$, however, each other components (from $\mathcal{C}_{\mathrm{b}}$) must contain a vertex from $\mathcal{Q}$. Two vertices of the same Q_i cannot be in different components thus $\omega(G-S) \leq c_{\mathrm{a}} + 2t + 3$.

By the assumption, $\mathcal{R} \subseteq S$. If $a_i \in \mathcal{C}_{\mathrm{a}}$ then $N(a_i) \subseteq S$, thus $q_i \in S$ for each i for which $a_i \in \mathcal{C}_{\mathrm{a}}$ therefore $|S| \geq |X| + |\mathcal{R}| + c_{\mathrm{a}}$ holds.

From this inequality, by the indirect assumption

$$(2t-\varepsilon)\,(c_{\mathrm{a}} + 2t + 3) > 2t + (2t-1)(2t+3)(2p+1) + c_{\mathrm{a}}$$

or equivalently

$$(2t-1-\varepsilon)\,(c_{\mathrm{a}} + 2t + 3) + 2t + 3 > 2t + (2t-1)(2t+3)(2p+1) \quad (4.6)$$

is obtained.

Using $2t-1-\varepsilon > 0$ and $c_{\mathrm{a}} \leq (2t+3)(2p+1)$ we obtain

$$(2t-1-\varepsilon)\,(2t+3)(2p+1) + (2t-\varepsilon)\,(2t+3) > 2t + (2t-1)(2t+3)(2p+1)$$

from (4.6) which is equivalent to

$$(2t-1-\varepsilon)(2p+1) + (2t-\varepsilon) > \frac{2t}{2t+3} + (2t-1)(2p+1).$$

This implies

$$\frac{t}{\varepsilon} > \frac{2t - \varepsilon - \frac{2t}{2t+3}}{2\varepsilon} - \frac{1}{2} > p$$

which is a contradiction since $p > \frac{t}{\varepsilon}$. ■

It should be mentioned that the special case of the above construction for $t = 1$ is very similar to a construction of Bauer, Broersma, van den Heuvel and Veldman [4]. As it was already mentioned Theorem 4.11 shows that Theorem 4.9 is sharp for any integer t. It is still not known if this is true for non–integer t values.

CONJECTURE 4.12 *For every $\varepsilon > 0$ rational number there exists a $(2t - \varepsilon)$–tough but non–t–edge–tough graph.*

It is worth mentioning that Theorem 4.9 implies

$$\frac{t_e(G)}{t(G)} \geq \frac{1}{2}$$

and Observation 4.7 implies

$$\frac{t_e(G)}{t(G)} \leq 1.$$

This suggests the following generalization of Conjecture 4.12 and 4.12.

CONJECTURE 4.13 *For every $\frac{1}{2} \leq c \leq 1$ rational number there exists a graph G for which*

$$\frac{t_e(G)}{t(G)} = c$$

holds.

It is also natural to make the following Conjecture as a version of Chvátal's Conjecture.

CONJECTURE 4.14 *There exists a t_1 such that every t_1–edge–tough graph is Hamiltonian.*

If such a t_1 exists then Theorem 4.9 implies that every $2t_1$-tough graph is Hamiltonian and therefore Chvátal's Conjecture is true. On the other hand, if Chvátal's Conjecture is true then Observation 4.7 implies that Conjecture 4.14 is true. Hence one may say that this Conjecture is equivalent to Chvátal's Conjecture.

4.4 1–edge–tough Graphs Have 2–factors

THEOREM 4.15 *Let G be a 1–edge–tough graph on $n \geq 3$ vertices. Then G has a 2–factor.*

PROOF: Suppose indirectly that G does not have a 2–factor. It will be proved that G is not 1–edge–tough.

If G does not have a 2–factor then, by Theorem 4.5, there exists a subset of vertices U such that

$$|U| < \frac{1}{2}\left(|K_U| + \kappa_U\right) \tag{4.7}$$

where K_U is the set of vertices having degree at most one in $G-U$ and where κ_U denotes the number of components of $G-U$ containing an odd number of vertices in K_U.

Let Z be the set of isolated vertices in $G-U$. The set of vertices with degree one in $G-U$ is denoted by A'. (That is, $K_U = Z \cup A'$.) Let $e_1, e_2, \dots, e_f$ be the edges of $G-U$ with both endvertices in A'. These edges are not adjacent to any other edge in $G-U$. The set of endvertices of these edges is denoted by F and $A := A' - F$. Let $C_1, C_2, \dots, C_c$, $(D_1, D_2, \dots, D_d)$ denote the components of $G-U-Z-F$ containing an even (odd) number of vertices of A.

Let p_i denote the number of those vertices in C_i which are adjacent to at least two vertices of A and r_i the number of those vertices in C_i which are adjacent to exactly one vertex of A. q_i and s_i are defined in the same way in D_i.

The indirect assumption (4.7) can be reformulated with these notations as

$$|A| + |Z| + 2f + d + |Z| > 2|U|$$

or equivalently

$$\frac{|A|}{2} + |Z| + f > |U| - \frac{d}{2}. \tag{4.8}$$

Now we give a lower bound on $|A|$. It is easy to see that C_i has at least $2p_i + r_i$ vertices of degree one. If r_i is odd then this lower bound is odd as well. Since C_i must have an even number of vertices of degree one $2p_i + r_i + 1$ is a better bound if r_i is even. Thus the number of vertices of degree one in C_i is at least

$$2p_i + r_i + 2\left(\frac{r_i}{2} - \left\lfloor \frac{r_i}{2} \right\rfloor\right) = 2p_i + 2r_i - 2\left\lfloor \frac{r_i}{2} \right\rfloor \tag{4.9}$$

where $\lfloor x \rfloor$ denotes the largest integer which is not larger than x. Similarly the number of vertices of degree one in D_i is at least

$$2q_i + s_i + 2\left(\frac{s_i+1}{2} - \left\lfloor \frac{s_i+1}{2} \right\rfloor\right) = 2q_i + 2s_i + 1 - 2\left\lfloor \frac{s_i+1}{2} \right\rfloor. \tag{4.10}$$

Summing (4.9) and (4.10) for all components C_i and D_i

$$|A| \geq 2\sum_1^c p_i + 2\sum_1^c r_i - 2\sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + 2\sum_1^d q_i + 2\sum_1^d s_i + d - 2\sum_1^d \left\lfloor \frac{s_i+1}{2} \right\rfloor$$

or equivalently

$$\frac{|A|}{2} \geq \sum_1^c p_i + \sum_1^c r_i - \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d q_i + \sum_1^d s_i + \frac{d}{2} - \sum_1^d \left\lfloor \frac{s_i+1}{2} \right\rfloor \tag{4.11}$$

is obtained.

The vertex set X and the edge set Y are defined next which will prove that the graph is not 1–edge–tough.

Let P denote the subset of those vertices of $G-U-Z-F$ whose vertices are adjacent to at least two vertices in A and Q the subset of vertices of $G-U-Z-F$ whose vertices are adjacent to exactly one vertex in A. Thus $|P| = \sum_1^c p_i + \sum_1^d q_i$ and $|Q| = \sum_1^c r_i + \sum_1^d s_i$. Let $X = U \cup P$.

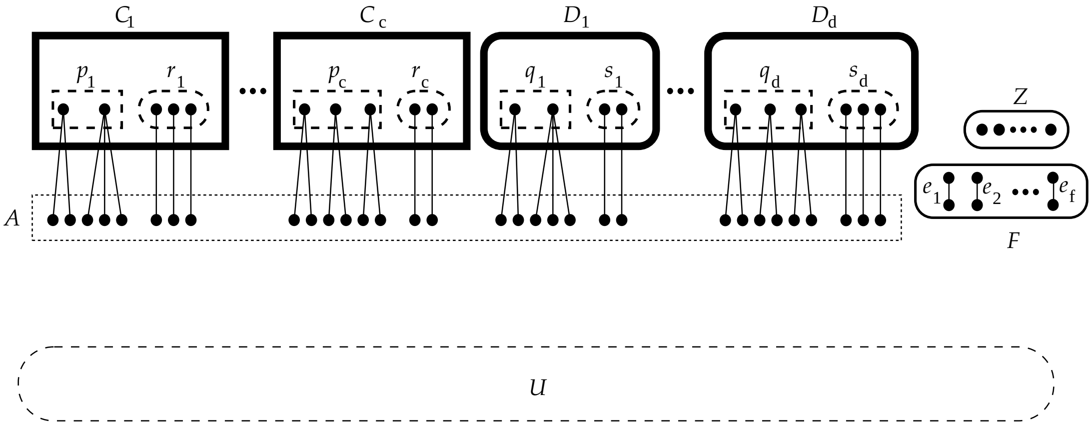

Fig. 4.3

Y is defined as follows. An edge of C_i or D_i belongs to Y if and only if it is incident neither to X nor to A. Formally $Y := E(G - X - A - Z - F)$. The components of the edge induced subgraph of $G(Y)$ are the components of $C_i - X$ and $D_i - X$ and the boundaries of these components are those vertices of C_i and D_i which are adjacent to exactly one vertex of A. The interiors of the components consist of those vertices which are not adjacent to A.

It is easy to see (Fig. 4.3) that $G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)$ is the union of the set of isolated vertices consisting of Z and those vertices of A which are adjacent to P and the set of independent edges $e_1, \ldots, e_f$ and the edges which connect A and Q. To finish the proof it remains to prove that this X and Y violate the inequality of Definition 4.6 for $t = 1$, namely the following inequality:

$$\omega\left(G - X - Y - \bigcup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) = |A| + |Z| + f > |X| + \sum_{i=1}^{n} \left\lfloor \frac{|\mathrm{bd}_{G-X}(Q_i)|}{2} \right\rfloor = s(X, Y; G).$$

In our case $\sum_1^n \left\lfloor \frac{|\mathrm{bd}_{G-X}(Q_i)|}{2} \right\rfloor \leq \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d \left\lfloor \frac{s_i}{2} \right\rfloor$. (Equality does not necessary hold because Q may have more than one component in any C_i or D_i.) Thus it is enough to prove

$$|A| + |Z| + f > |U| + \sum_1^c p_i + \sum_1^d q_i + \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d \left\lfloor \frac{s_i}{2} \right\rfloor . \tag{4.12}$$

Summing (4.8) and (4.11) we obtain

$$|A| + |Z| + f > |U| - \frac{d}{2} + \sum_1^c p_i + \sum_1^c r_i - \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d q_i + \sum_1^d s_i + \frac{d}{2} - \sum_1^d \left\lfloor \frac{s_i + 1}{2} \right\rfloor . \tag{4.13}$$

Since $x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor$ and $x \geq 2 \left\lfloor \frac{x}{2} \right\rfloor$ for any non negative integer x it is easy to see that

$$\sum_1^c r_i - \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d s_i - \sum_1^d \left\lfloor \frac{s_i + 1}{2} \right\rfloor \geq \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d \left\lfloor \frac{s_i}{2} \right\rfloor$$

holds. (4.12) can be obtained from (4.13) using this inequality. ■

The results in [34] suggest that the previous Theorem may be generalized.

CONJECTURE 4.16 *Let G be a t–edge–tough graph on $n \geq 2t+1$ vertices where t is a positive integer. Then G has a $2t$–factor.*

If this were true, it would be a sharpening of Theorem 4.2.

Chapter 5

Hamiltonian chains

5.1 Introduction to Hamiltonian Chains

Let $\mathcal{H}$ be a k–uniform hypergraph on the vertex set $V(\mathcal{H}) = \{v_1, v_2, \ldots, v_n\}$ where $n > k$. v_{n+x} with $x \geq 0$ denotes the same vertex as v_x for simplicity of notation if not stated otherwise. The set of the edges, k–element subsets of V, is denoted by $\mathcal{E}(\mathcal{H}) = \{E_1, E_2, \ldots, E_m\}$. We will write simply V for $V(\mathcal{H})$ and $\mathcal{E}$ for $\mathcal{E}(\mathcal{H})$ if no confusion can arise. The complete k–uniform hypergraph on n vertices is denoted by $\mathcal{K}_n^{(k)}$.

Bermond et al. in [11] defined the Hamiltonian cycle in hypergraphs and extended different Theorems of Hamiltonian graph theory to hypergraphs.

Definition 5.1 [11] *A $(v_1, v_2, \ldots, v_n)$ cyclic sequence of the vertex set is called a hypergraph Hamiltonian cycle iff for every $1 \leq i \leq n$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}\} \subseteq E_j$.*

Since an ordinary graph is a 2-uniform hypergraph, Definition 5.1 gives the definition of the Hamiltonian cycle in ordinary graphs for $k = 2$.

The earliest result about the Hamiltonian cycle is the following Theorem of Dirac.

Theorem 5.2 [30] *If G is a graph on $n \geq 3$ vertices such that $d(v) \geq \frac{1}{2}n$ for every vertex v in G, then G contains a Hamiltonian cycle, where $d(v)$ denotes the degree of v.*

This Theorem is best possible in the sense that the bound $\frac{1}{2}n$ cannot be relaxed without destroying the conclusion of the Theorem. This is shown by the complete bipartite graphs $K_{n,n+1}$ $(n \geq 1)$.

The *total degree* of a vertex v in a hypergraph, by definition, is the number of edges containing v. It is denoted by $\delta(v)$. Bermond et al. proved a Dirac type Theorem. It is worth noting that this does not give the best possible result for $k = 2$.

Theorem 5.3 [11] *Let $\mathcal{H}$ be a k–uniform hypergraph on $n \geq k+1$ vertices. If $\delta(v) \geq \binom{n-2}{k-1} + k - 1$ holds for every vertex v of the hypergraph then $\mathcal{H}$ contains a Hamiltonian cycle.*

Definition 5.1 is slightly modified to obtain the definition of the *Hamiltonian chain* as an other generalization of the Hamiltonian cycle. Generalizations of path, cycle and Hamiltonian path are defined analogously.

Definition 5.4 *A cyclic ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set is called a Hamiltonian chain iff for every $1 \leq i \leq n$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} = E_j$. An ordering $(v_1, v_2, \ldots, v_{l+1})$ of a subset of the vertex set is called an open chain of length l between v_1 and v_{l+1} iff for every $1 \leq i \leq l-k+2$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} = E_j$. An open chain of length $n-1$ is an open Hamiltonian chain. A* *cyclic ordering $(v_1, v_2, \ldots, v_l)$ of a subset of the vertex set is called a chain of length l iff for every $1 \leq i \leq l$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} = E_j$. (Now v_{l+x} denotes the same vertex as v_x).*

This definition was motivated by two different problems. In [29] a similar, however more complicated problem arose in connection with database theory. The other motivating problem concerns the powers of Hamiltonian cycles. The kth power of a cycle C is obtained by joining every pair of vertices with distance at most k in C. Seymour conjectured in [78] that G contains the kth power of a Hamiltonian cycle if $d(v) \geq \frac{k}{k+1}n$ holds for every vertex in the graph. Let us transform a k–uniform hypergraph to a simple graph by replacing every hyperedge by a complete graph on the same k vertices and replacing multiple edges with a single edge. In this way a Hamiltonian chain is transformed to the $(k-1)$th power of a Hamiltonian cycle. However, it is not true that if the graph contains the $(k-1)$th power of a Hamiltonian cycle then the original hypergraph contains a Hamiltonian chain. Therefore the results in [37] about the square of the Hamiltonian cycle does not imply anything for Hamiltonian chains. On the contrary, our result implies a result for the square of a Hamiltonian cycle with a different degree condition.

There are many more possibilities to modify the definitions, especially if $k \geq 4$. Only one more of them will be investigated in the present thesis.

Definition 5.5 *Let $\mathcal{H}$ be a 4–uniform hypergraph. A cyclic ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set is called a Hamiltonian net iff for every $1 \leq i \leq n$ and $1 \leq j \leq n$ for which $|i-j| \geq 2$ there exists an edge E_m of $\mathcal{H}$ such that $\{v_i, v_{i+1}, v_j, v_{j+1}\} = E_m$. ($v_{n+x}$ with $x \geq 0$ denotes the same vertex as v_x.)*

In order to give Dirac type results the definition of the degree must be generalized, too. It is defined now in full generality, however, only some special cases will be used.

Definition 5.6 *The degree of a fixed l–tuple of distinct vertices, $\{v_1, v_2, \ldots, v_l\}$, in a k–uniform hypergraph is the number of edges of the hypergraph containing all $\{v_1, v_2, \ldots, v_l\}$. It is denoted by $d_k(v_1, v_2, \ldots, v_l)$.*

Finally $\lfloor x \rfloor$ denotes the largest integer which is not larger than x and $\lceil x \rceil$ is the smallest integer which is not smaller than x, as usual.

A Dirac type Theorem is proved for 3–uniform hypergraphs in Section 5.2, a natural extremal question is discussed in Section 5.3. Section 5.4 contains a construction which shows that Hamiltonian nets differ from Hamiltonian chains very much.

5.2 Hamiltonian Chains in 3–uniform Hypergraphs

Theorem 5.7 *Let $\mathcal{H}$ be a 3–uniform hypergraph on $n \geq 3$ vertices. If*

$$d_3(x,y) > \left(\sqrt{47} - 6\right) n + 26 - \frac{353}{94}\sqrt{47} = c_1 n + c_2 \approx 0.8556 \cdot n + 0.2548 \tag{5.1}$$

holds for every pair of vertices x, y then there is a Hamiltonian chain in $\mathcal{H}$.

Proof: Suppose indirectly that there exists a hypergraph which satisfies the condition but does not contain a Hamiltonian chain. If there is more than one such hypergraph then let $\mathcal{H}$

be a maximal one. This means that if any new edge is added to $\mathcal{H}$ then the new hypergraph contains a Hamiltonian chain. In particular suppose that if the edge $\{v_{n-1}, v_n, v_1\}$ is added to $\mathcal{H}$ then $(v_1, v_2, \ldots, v_n)$ becomes a Hamiltonian chain. Thus, by definition, $P = (v_1, v_2, \ldots, v_n)$ is an open Hamiltonian chain from v_1 to v_n. It will be shown that $\mathcal{H}$ contains a Hamiltonian chain, contradicting our assumption. In fact, we show that there exist an i and a j where $3 \leq i \leq n-2$, $3 \leq j \leq n-2$, $j < i$ and $|i-j| \geq 2$ such that either $C = (v_1, v_2, \ldots v_{j-1}, v_{i-1}, v_{i-2}, \ldots, v_{j+1}, v_{i+1}, v_{i+2}, \ldots, v_{n-1}, v_n, v_i, v_j)$ or $C' = (v_1, v_2, \ldots v_{j-1}, v_{i-1}, v_{i-2}, \ldots, v_{j+1}, v_{i+1}, v_{i+2}, \ldots, v_{n-1}, v_n, v_j, v_i)$ is a Hamiltonian chain (see Fig. 5.1).

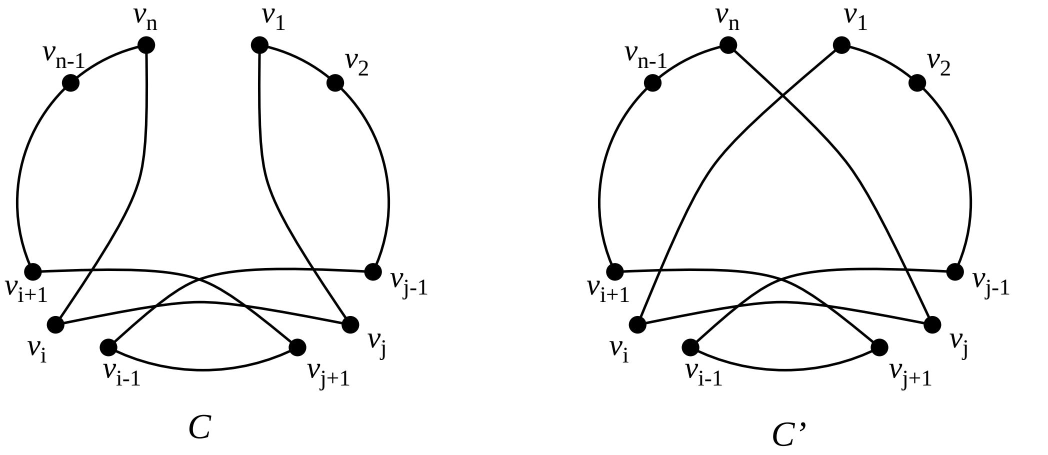

Fig. 5.1

To prove that C is a Hamiltonian chain it must be shown that the following set of edges are in $\mathcal{H}$ since the rest of the required edges are edges of P.

$$\begin{aligned}
A(i,j) &= \{\{v_{n-1}, v_n, v_i\}, \{v_1, v_2, v_j\}\}, \\
L(i,j) &= \{\{v_{i+2}, v_{i+1}, v_{j+1}\}, \{v_{i-2}, v_{i-1}, v_{j-1}\}, \{v_n, v_i, v_j\}\}, \\
R(i,j) &= \{\{v_{j-2}, v_{j-1}, v_{i-1}\}, \{v_{j+1}, v_{j+2}, v_{i+1}\}, \{v_1, v_j, v_i\}\}.
\end{aligned}$$

The corresponding sets for C' are

$$\begin{aligned}
A'(i,j) &= A(j,i), \\
L'(i,j) &= L(i,j), \\
R'(i,j) &= R(i,j).
\end{aligned}$$

Four auxiliary graphs are defined to make the proof more clear. Let G_1 be a directed bipartite graph with color classes $X = \{x_3, \ldots, x_{n-2}\}$ and $Y = \{y_3, \ldots, y_{n-2}\}$. Both x_i and y_i correspond to the vertex v_i of $\mathcal{H}$. The directed edge (x_i, y_j) is in $E(G_1)$ iff $L(i,j) \subseteq \mathcal{E} = \mathcal{E}(\mathcal{H})$ and the directed edge (y_j, x_i) is in $E(G_1)$ iff $R(i,j) \subseteq \mathcal{E}$. Note that in this way there are no edges of the form (x_i, y_i) or (y_i, x_i). A lower bound is given for the out degree of a vertex x_i which is denoted by $d^{\text{out}}_{G_1}(x_i)$. This out degree is the number of those indices j $(3 \leq j \leq n-2)$ for which $L(i,j) \subseteq \mathcal{E}$. Note that $|i-j| \geq 2$ is not supposed here but $i \neq j$ is a consequence of the definition. Let $L_1(i)$ be the set of indices j for which $\{v_{i+2}, v_{i+1}, v_{j+1}\} \in \mathcal{E}$ and $3 \leq j \leq n-2$. $L_2(i)$ and $L_3(i)$ is defined similarly for $\{v_{i-2}, v_{i-1}, v_{j-1}\}$ and $\{v_n, v_i, v_j\}$. $d_3(v_{i+1}, v_{i+2}) > c_1 n + c_2$ implies $|L_1(i)| > c_1 n + c_2 - 4$, since $L_1(i)$ cannot contain $1, 2, n$ and $n-1$. The same is true for $|L_2(i)|$ because $d_3(v_{i-1}, v_{i-2}) > c_1 n + c_2$. $|L_3(i)| > c_1 n + c_2 - 4$ holds as well since $d_3(v_n, v_i) > c_1 n + c_2$.

Clearly $|L_1(i) \cap L_2(i) \cap L_3(i)|$ must be estimated to give the lower bound for the out degree. Considering the complements on $\{3, \ldots, n-2\}$ we obtain

$$d_{G_1}^{\text{out}}(x_i) > 3c_1 n + 3c_2 - 2n - 4 \tag{5.2}$$

holds for every $x_i \in X$. It can be proved similarly that

$$d_{G_1}^{\text{out}}(y_i) > 3c_1 n + 3c_2 - 2n - 4 \tag{5.3}$$

holds for every $y_i \in Y$.

Now a simple graph G_2 is defined on the vertex set $\{p_3, \ldots, p_{n-2}\}$. The edge $\{p_i, p_j\}$ is in $E(G_2)$ iff $(x_i, y_j) \in E(G_1)$ and $(y_j, x_i) \in E(G_1)$. Using (5.2) and (5.3) it is easy to show that

$$\begin{aligned} |E(G_2)| >& 2\,(3c_1 n + 3c_2 - 2n - 4)\,(n-4) - (n-4)^2 + n - 4 = \\ =& (6c_1 n + 6c_2 - 5n - 3)\,(n-4) \end{aligned} \tag{5.4}$$

since the number of pairs $\{x_i, y_j\}$ for which $i \neq j$ is $(n-4)^2 - (n-4)$. $|E(G_2)|$ is the number of pairs i, j for which $L(i,j) \cup R(i,j) \subseteq \mathcal{E}$ and $3 \leq i \leq n-2$, $3 \leq j \leq n-2$. Note that $L(i,j) \cup R(i,j)$ is symmetric in i and j thus there is no need for G_2 to be directed.

The third auxiliary graph G_3 is defined on the same vertex set, $\{p_3, \ldots, p_{n-2}\}$, as G_2. The edge $\{i, j\}$ $(i \neq j)$ is in $E(G_3)$ iff $A(i,j) \subseteq \mathcal{E}$ or $A'(i,j) \subseteq \mathcal{E}$. Note that this condition is symmetric in i and j. There are more than $c_1 n + c_2$ indices j for which $(v_1, v_2, v_j) \in \mathcal{E}$, however, this may include the cases $j = n-1$ and $j = n$. There are more than $c_1 n + c_2$ indices i for which $(v_{n-1}, v_n, v_i) \in \mathcal{E}$, however, this includes the case $i = 2$. (It does not include the case $i = 1$ because $\{v_{n-1}, v_n, v_1\} \notin \mathcal{E}$ by our assumption.) Therefore

$$|E(G_3)| > \frac{1}{2}(c_1 n + c_2 - 2)(c_1 n + c_2 - 2) \tag{5.5}$$

holds because every edge is counted at most twice and $i \neq j$ must hold.

Let $V(G_4) = V(G_3)(= V(G_2))$ and $E(G_4) = E(G_2) \cap E(G_3)$. It is easy to see that

$$|E(G_4)| \geq |E(G_2)| + |E(G_3)| - \binom{n-4}{2}$$

holds. Finally we have to take into consideration the condition $|i - j| \geq 2$ as well. $i = j$ was excluded, the number of eventual edges of the form $\{p_i, p_{i+1}\}$ should be subtracted. Therefore for the number N of pairs i, j which satisfy $A(i,j) \subseteq \mathcal{E}$ or $A'(i,j) \subseteq \mathcal{E}$ and $L(i,j) \cup R(i,j) \subseteq \mathcal{E}$, $3 \leq i \leq n-2$, $3 \leq j \leq n-2$ and $|i - j| \geq 2$ we obtain

$$N > (6c_1 n + 6c_2 - 5n - 3)\,(n-4) + \frac{1}{2}(c_1 n + c_2 - 2)(c_1 n + c_2 - 2) - \binom{n-4}{2} - (n-5) \tag{5.6}$$

using (5.4) and (5.5). Suitably arranging (5.6)

$$N > n^2 \left(\frac{1}{2}c_1^2 + 6c_1 - \frac{11}{2}\right) + n\left(c_1 c_2 + 6c_2 - 26c_1 + \frac{41}{2}\right) + \frac{1}{2}c_2^2 - 26c_2 + 9$$

is obtained. The values of c_1 and c_2 are chosen so that the coefficient of n^2 and n is zero and the constant term is positive (in fact, it is > 2). This implies that $N \geq 1$, hence there exists a suitable pair i, j. All conditions are symmetric in i and j therefore we can assume that $i > j$ so C is a Hamiltonian chain if $A(i,j) \subseteq \mathcal{E}$ and C' is a Hamiltonian chain if $A'(i,j) = A(j,i) \subseteq \mathcal{E}$. This contradicts the assumption that $\mathcal{H}$ is a counter example. ■

The bound given in Theorem 5.7 is probably not best possible. However, it is quite easy to show that this bound cannot be lower than $\lfloor \frac{n-k+2}{2} \rfloor$ for k–uniform hypergraphs.

DEFINITION 5.8 *Let $\mathcal{S}_n^{(k)}$ be a k–uniform hypergraph on the vertex set $V\left(\mathcal{S}_n^{(k)}\right) = X \cup Y$ where $X \cap Y = \emptyset$, $|X| = \left\lfloor \frac{n}{2} \right\rfloor$ and $|Y| = \left\lceil \frac{n}{2} \right\rceil$. ($X$ and Y are shortly called the sides of the hypergraph.) $\mathcal{E}\left(\mathcal{S}_n^{(k)}\right)$ contains every k–element subset E of $X \cup Y$ for which $E \cap X \neq \left\lfloor \frac{k}{2} \right\rfloor$.*

THEOREM 5.9 *$\mathcal{S}_n^{(k)}$ does not contain a Hamiltonian chain and for every $(k-1)$–element subset $\{p_1, \ldots p_{k-1}\}$ of the vertex set*

$$d_k(p_1, \ldots, p_{k-1}) \geq \left\lfloor \frac{n-k}{2} \right\rfloor$$

holds.

PROOF: Suppose that $(v_1, \ldots, v_n)$ is a Hamiltonian chain in $\mathcal{S}_n^{(k)}$. Let $E_i = \{v_i, \ldots, v_{i+k-1}\}$ and $x_i = |E_i \cap X|$. It is easy to see that $x_{i+1} = x_i$ iff v_i and v_{i+k} are on the same side of the hypergraph and $|x_{i+1} - x_i| = 1$ iff they are in different sides. The equation

$$\sum_{i=1}^{n} x_i = k \left\lfloor \frac{n}{2} \right\rfloor \tag{5.7}$$

holds because each vertex of X is counted once for each edge containing it, thus exactly k times. Using (5.7) we conclude that there is an index q such that $x_q \geq \left\lceil \frac{k}{n} \left\lfloor \frac{n}{2} \right\rfloor \right\rceil$ and an index r such that $x_r \leq \left\lfloor \frac{k}{n} \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor$. On the other hand, the numbers x_i form a sequence where two consecutive members differ in at most one. Hence every integer in the closed interval $I = \left[\left\lfloor \frac{k}{n} \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor, \left\lceil \frac{k}{n} \left\lfloor \frac{n}{2} \right\rfloor \right\rceil\right]$ must occur in the sequence. If n is even then $I = \left[\left\lfloor \frac{k}{2} \right\rfloor, \left\lceil \frac{k}{2} \right\rceil\right]$ and if n is odd then $I = \left[\left\lfloor \frac{k(n-1)}{2n} \right\rfloor, \left\lceil \frac{k(n-1)}{2n} \right\rceil\right]$ thus clearly $\left\lfloor \frac{k}{2} \right\rfloor \in I$ since $k < n$. It follows that there is an edge in the hypergraph which contains exactly $\left\lfloor \frac{k}{2} \right\rfloor$ vertices of X. But there is no such edge in $\mathcal{S}_n^{(k)}$ by definition, a contradiction.

To prove the degree condition suppose first that $|\{v_1, \ldots, v_{k-1}\} \cap X| = \left\lfloor \frac{k}{2} \right\rfloor$ and

$$|\{v_1, \ldots, v_{k-1}\} \cap Y| = \left\lceil \frac{k}{2} \right\rceil - 1.$$

Then adding any vertex to $\{v_1, \ldots, v_{k-1}\}$ from X yields an edge of the hypergraph therefore $d_k(v_1, \ldots, v_{k-1}) \geq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor$ holds in this case.

Suppose now that $|\{v_1, \ldots, v_{k-1}\} \cap X| = \left\lfloor \frac{k}{2} \right\rfloor - 1$ and $|\{v_1, \ldots, v_{k-1}\} \cap Y| = \left\lceil \frac{k}{2} \right\rceil$. Then adding any vertex to $\{v_1, \ldots, v_{k-1}\}$ from Y yields an edge of the hypergraph therefore $d_k(v_1, \ldots, v_{k-1}) \geq \left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil$ holds in this case.

For any other $(k-1)$–element subset $d_k(v_1, \ldots, v_{k-1}) \geq n - k + 1$. Therefore

$$d_k(v_1, \ldots, v_{k-1}) \geq \min\left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil\right) = \left\lfloor \frac{n-k}{2} \right\rfloor$$

holds for any $(k-1)$–element subset of the vertex set. ■

5.3 Many Edges and No Hamiltonian Chain

A natural extremal question about the Hamiltonian cycle is that how may edges a graph without a Hamiltonian cycle can contain. The extremal case is a complete graph on $(n-1)$ vertices completed by a vertex of degree one. It is trivial to prove that this graph does not contain a Hamiltonian cycle, but it is not straightforward to prove that this is the extremal case.

The same question is raised for hypergraph and Hamiltonian chain. However, even determining the extremal case looks more difficult.

THEOREM 5.10 *If $\mathcal{H}$ is a k–uniform hypergraph on n vertices satisfying*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{n-1}{n}\binom{n}{k} \tag{5.8}$$

then $\mathcal{H}$ contains a Hamiltonian chain.

PROOF: Let m denote the number of missing edges (the k–element subsets which are not edges of $\mathcal{H}$). By (5.8) we obtain

$$m < \binom{n}{k} - \frac{n-1}{n}\binom{n}{k} = \frac{1}{n}\binom{n}{k}.$$

It is a simple matter to prove that there are $\frac{1}{2}(n-1)!$ different Hamiltonian chains in $\mathcal{K}_n^{(k)}$. It is also trivial that $\frac{1}{2}k!(n-k)!$ of these contain a fixed edge. Obviously, if a Hamiltonian chain in $\mathcal{K}_n^{(k)}$ contains an edge which is not an edge of $\mathcal{H}$ then it is not a Hamiltonian chain in $\mathcal{H}$. Thus the number of Hamiltonian chains in $\mathcal{H}$ is

$$\geq \frac{1}{2}(n-1)! - \frac{1}{2}k!(n-k)!m > 0,$$

our claim is proved. ■

At an earlier stage we found a construction with a fairly large number of edges containing no Hamiltonian chain. Later better constructions were found but surprisingly this is still the best one in the special case when $k=3$ and $3|(n-1)$. Therefore this construction may be of some interest in general, too.

THEOREM 5.11 *For any $n \geq 6$ and $3 \leq k \leq n-3$ there exists a k–uniform hypergraph $\mathcal{H}(n,k)$ which does not contain a Hamiltonian chain and satisfies*

$$|\mathcal{E}(\mathcal{H}(n,k))| = \binom{n-1}{k^*} + q\binom{2k^*-3}{k^*-1} + \binom{r}{k^*-1} \tag{5.9}$$

where q and $r < q$ are the unique nonnegative integers satisfying $n-1 = q(2k^*-3)+r$ and

$$k^* = \begin{cases} k & \text{if } k \leq \frac{n}{2} \\ n-k & \text{if } k > \frac{n}{2}. \end{cases}$$

PROOF: We first define the hypergraph $\mathcal{H}(n,k)$ only if $k \le \frac{n}{2}$, thus $k^* = k$ in this part of the proof. Let us partition the vertex set $V - \{v_n\}$ into $q+1$ sets $V_1, \ldots, V_{q+1}$ such that $|V_1| = \ldots = |V_q| = 2k-3$ and $|V_{q+1}| = r$. The partition have at least two nonempty sets since $n-1 > 2k-3$ holds if $k \le \frac{n}{2}$. The edge set $\mathcal{E}(\mathcal{H}(n,k))$ consists of all k–element subset of $V - \{v_n\}$, denoted by $\mathcal{E}_1(\mathcal{H}(n,k))$, and all $(k-1)$–element subset of V_i $(1 \le i \le q+1)$ extended with the vertex v_n, denoted by $\mathcal{E}_2(\mathcal{H}(n,k))$. It follows immediately that (5.9) holds.

To prove that $\mathcal{H}(n,k)$ does not contain a Hamiltonian chain suppose the contrary. Without loss of generality we can assume that this Hamiltonian chain is $(v_1, \ldots, v_n)$. The *restriction* of the Hamiltonian chain on $V - \{v_n\}$ is a $(k-1)$–uniform open chain of length $2k-3$ connecting v_{n-k+1} and v_{k-1}. On the other hand the restriction of $\mathcal{E}_2(\mathcal{H}(n,k))$ on $V - \{v_n\}$ consists of q copies of $K_{2k-3}^{(k-1)}$ and a $K_r^{(k-1)}$ which clearly cannot contain an open chain of length $2k-3$, a contradiction.

Now the hypergraph $\mathcal{H}(n,k)$ is defined if $k > \frac{n}{2}$. $\mathcal{H}(n,k)$ is obtained from $\mathcal{H}(n,n-k)$ (which is already defined) setting $V(\mathcal{H}(n,k)) = V(\mathcal{H}(n,n-k))$ and replacing each edge $E_i \in \mathcal{H}(n,n-k)$ by the edge $V(\mathcal{H}(n,n-k)) - E_i$. It is clear that (5.9) is satisfied. On the other hand it follows immediately from the definition of the Hamiltonian chain that $\mathcal{H}(n,k)$ contains a Hamiltonian chain if and only if $\mathcal{H}(n,n-k)$ does. Since $\mathcal{H}(n,n-k)$ does not contain a Hamiltonian chain, the proof is complete. ∎

This was improved by a construction of Tuza [81] giving

$$|\mathcal{E}| = \binom{n-1}{k} + (1-o(1))\binom{n-2}{k-2}.$$

This construction led us to the following construction what we conjecture to be the best possible if $k \ge 4$. The construction of Theorem 5.11 has the same number of edges if $k = 3$ and $3 \nmid (n-1)$ but has one more edge if $k = 3$ and $3|(n-1)$.

THEOREM 5.12 *For any $n \ge 4$ and $2 \le k \le n-2$ there exists a k–uniform hypergraph $\mathcal{H}(x,y)$ which does not contain a Hamiltonian chain and satisfies*

$$|\mathcal{E}(\mathcal{H}(x,y))| = \binom{n-1}{k} + \binom{n-2}{k-2} = \binom{n}{k} - \binom{n-2}{k-1}. \tag{5.10}$$

PROOF: Let x and y be two distinct vertices of the vertex set. $\mathcal{E}(\mathcal{H}(x,y))$ consists of all k element subsets of the vertex set except those which contain x but do not contain y. In other words, if an edge contains x it must contain y, too. It is clear that (5.10) holds.

To prove that $\mathcal{H}(x,y)$ does not contain a Hamiltonian chain suppose the contrary. Without loss of generality we can assume that this Hamiltonian chain is $(v_1, \ldots, v_n)$ and that $x = v_1$. Let $y = v_i$. It will be shown that there is an edge of the Hamiltonian chain which contains x but does not contain y. If $k+1 \le i \le n$ then the edge $(v_1, \ldots v_k)$ is such an edge. $2 \le i \le k$ then the edge $(v_{n-k+i}, \ldots, v_{i-1})$ has this property proving our claim. ∎

5.4 Hamiltonian Nets

The following Theorem shows that a lower bound cn with $c < 1$ for the degrees of triples is not sufficient to force the existence of a Hamiltonian net.

THEOREM 5.13 *For any given integer $n \geq 6$ there exists a 4-uniform hypergraph $\mathcal{H}$ on n vertices which does not contain a Hamiltonian net and*

$$d_4(x,y,z) \geq \begin{cases} n-9 & \text{if } n \text{ is odd} \\ n-12 & \text{if } n \text{ is even} \end{cases}$$

holds for any x, y, z triple of vertices.

PROOF: The construction of $\mathcal{H}$ is given first for odd n. To shorten notation let $r = \frac{n-1}{2}$. It is a well known fact that one can partition the edges of the complete graph $K_n^{(2)}$ into Hamiltonian cycles so let $E\left(K_n^{(2)}\right) = H_1 \cup \dots \cup H_r$ where $H_1, \dots, H_r$ are the edge sets of edge disjoint Hamiltonian cycles. $\mathcal{H}$ is defined on the vertex set of $K_n^{(2)}$. A four element subset of the vertex set is a hyperedge of $\mathcal{H}$ iff these vertices do not span two nonadjacent edges of the same H_i.

Suppose indirectly that $\mathcal{H}$ contains a Hamiltonian net. The cyclic ordering can be considered as a Hamiltonian cycle C in $K_n^{(2)}$ which has $n = 2r + 1$ edges, of course. Since the edge set is partitioned into r Hamiltonian cycles there is an H_i which has at least 3 common edges with C and thus there are at least two of them which are nonadjacent. The four end vertices of these two edges does not form a hyperedge of $\mathcal{H}$ by definition, a contradiction.

We prove now that the degree condition holds for $\mathcal{H}$. Let x, y, z be arbitrary distinct vertices of $\mathcal{H}$. An upper bound will be given for the number of those vertices v for which $\{x, y, z, v\} \notin \mathcal{E}(\mathcal{H})$. Each of the three pairs $\{x, y\}, \{x, z\}$ and $\{y, z\}$ is an edge of $K_n^{(2)}$ so each of them is contained in exactly one H_i. Without loss of generality we may assume that $\{x, y\} \in H_1$. z is also a vertex of the Hamiltonian cycle formed by the edges of H_1. It is clear that $\{x, y, z, v\} \notin \mathcal{E}(\mathcal{H})$ iff v is a neighbor of z on this Hamiltonian cycle. If z is not a neighbor of x or y on this Hamiltonian cycle then there are two such vertices, otherwise only one. In the same manner one can prove that there may be at most two other choices for v along the Hamiltonian cycle containing $\{x, z\}$ and two other along the one containing $\{y, z\}$. Thus there are at most 6 "bad" vertices v out of the $n - 3$ possible choices proving that $d_4(x, y, z) \geq n - 9$ always holds in this case.

The proof is similar when n is even. Now let $r = \frac{n-2}{2}$. Obviously it is not possible to partition the edge set of $K_n^{(2)}$ into Hamiltonian cycles, however, it is also well known that it can be partitioned into r Hamiltonian cycles and a perfect matching. We define a new partition by rearranging this. Let $E\left(K_n^{(2)}\right) = H_1' \cup \dots \cup H_r' \cup \{e\}$ where each H_i' is the union of a Hamiltonian cycle and one of the edges from the perfect matching (each H_i' is a Hamiltonian cycle with a chord) and $\{e\}$ is the remaining edge of the perfect matching. $H_1' \cup \dots \cup H_r'$ can also be considered as a partition of $K_n^{(2)} - \{e\}$ into r edge disjoint sets. $\mathcal{H}$ is defined on the vertex set of $K_n^{(2)}$. A four element subset of the vertex set is a hyperedge of $\mathcal{H}$ iff these vertices do not span two nonadjacent edges of the same H_i'.

Suppose indirectly that $\mathcal{H}$ contains a Hamiltonian net. The cyclic ordering can be considered as a Hamiltonian cycle C in $K_n^{(2)}$ which have $n = 2r + 2$ edges in this case. It may contain

e but it has $2r+1$ edges different from e. Since the edge set is partitioned into r sets, there is an H_i' which has at least 3 common edges with C. There are at least two of them which are nonadjacent because otherwise they should form a triangle which is impossible, since they are edges of a Hamiltonian cycle. The four end vertices of these two edges do not form a hyperedge of $\mathcal{H}$ by definition, a contradiction.

The proof for the degree condition is similar to the previous case, too. The only difference is that if $\{x, y\}$ is an edge of H_1' then it is possible that z has three neighbors if z is an endvertex of the chord. This gives at most 9 "bad" choices of v out of the $n-3$ possibilities. ■

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