Hamiltonian Chains in Hypergraphs

Gyula Y. Katona*
Mathematical Institute of the
Hungarian Academy of Sciences

Hal Kierstead
Arizona State University
Tempe

March 1, 2002

1 Introduction to Hamiltonian Chains

Let $\mathcal{H}$ be a k–uniform hypergraph on the vertex set $V(\mathcal{H}) = \{v_1, v_2, \ldots, v_n\}$ where $n > k$. v_{n+x} with $x \geq 0$ denotes the same vertex as v_x for simplicity of notation if not stated otherwise. The set of the edges, k–element subsets of V, is denoted by $\mathcal{E}(\mathcal{H}) = \{E_1, E_2, \ldots, E_m\}$. We will write simply V for $V(\mathcal{H})$ and $\mathcal{E}$ for $\mathcal{E}(\mathcal{H})$ if no confusion can arise. The complete k–uniform hypergraph on n vertices is denoted by $\mathcal{K}_n^{(k)}$.

Bermond et al. in [1] defined the hamiltonian cycle in hypergraphs and extended different Theorems of hamiltonian graph theory to hypergraphs.

Definition 1 [1] *A $(v_1, v_2, \ldots, v_n)$ cyclic sequence of the vertex set is called a* hypergraph hamiltonian cycle *iff for every $1 \leq i \leq n$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}\} \subseteq E_j$.*

Since an ordinary graph is a 2-uniform hypergraph, Definition 1 gives the definition of the hamiltonian cycle in ordinary graphs for $k = 2$.

The earliest result about the hamiltonian cycle is the following Theorem of Dirac.

Theorem 2 [3] *If G is a graph on $n \geq 3$ vertices such that $d(v) \geq \frac{1}{2}n$ for every vertex v in G, then G contains a hamiltonian cycle, where $d(v)$ denotes the degree of v.*

This Theorem is best possible in the sense that the bound $\frac{1}{2}n$ cannot be relaxed without destroying the conclusion of the Theorem. This is shown by the complete bipartite graphs $K_{n,n+1}$ $(n \geq 1)$.

The *total degree* of a vertex v in a hypergraph, by definition, is the number of edges containing v. It is denoted by $\delta(v)$. Bermond et al. proved a Dirac type Theorem. It is worth noting that this does not give the best possible result for $k = 2$.

Theorem 3 [1] *Let $\mathcal{H}$ be a k–uniform hypergraph on $n \geq k+1$ vertices. If $\delta(v) \geq \binom{n-2}{k-1} + k - 1$ holds for every vertex v of the hypergraph then $\mathcal{H}$ contains a hamiltonian cycle.*

Definition 1 is slightly modified to obtain the definition of the *hamiltonian chain* as an other generalization of the hamiltonian cycle. Generalizations of path, cycle and hamiltonian path are defined analogously.

Definition 4 *A cyclic ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set is called a* hamiltonian chain *iff for every $1 \leq i \leq n$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} = E_j$. An ordering $(v_1, v_2, \ldots, v_{l+1})$ of a subset of the vertex set is called an* open chain of length l *between v_1 and v_{l+1} iff for every $1 \leq i \leq l-k+2$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} = E_j$. An open chain of length $n-1$ is an* open hamiltonian chain*. A cyclic ordering $(v_1, v_2, \ldots, v_l)$ of a subset of the vertex set is called a* chain of length l *iff for every $1 \leq i \leq l$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} = E_j$. (Now v_{l+x} denotes the same vertex as v_x).*

*This work was partially supported by the Hungarian National Foundation for Scientific Research, OTKA Grant Numbers F 014919 and T 014302

This definition was motivated by two different problems. In [2] a similar, however more complicated problem arose in connection with database theory. The other motivating problem concerns the powers of hamiltonian cycles. The kth power of a cycle C is obtained by joining every pair of vertices with distance at most k in C. Seymour conjectured in [6] that G contains the kth power of a hamiltonian cycle if $d(v) \geq \frac{k}{k+1}n$ holds for every vertex in the graph. Let us transform a k–uniform hypergraph to a simple graph by replacing every hyperedge by a complete graph on the same k vertices and replacing multiple edges with a single edge. In this way a hamiltonian chain is transformed to the $(k-1)$th power of a hamiltonian cycle. However, it is not true that if the graph contains the $(k-1)$th power of a hamiltonian cycle then the original hypergraph contains a hamiltonian chain. Therefore the results in [4, 5] about the square of the hamiltonian cycle does not imply anything for hamiltonian chains. On the contrary, our result implies a result for the square of a hamiltonian cycle with a different degree condition.

There are many more possibilities to modify the definitions, especially if $k \geq 4$. Only one more of them will be investigated in the present thesis.

Definition 5 *Let $\mathcal{H}$ be a 4–uniform hypergraph. A cyclic ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set is called a* hamiltonian net *iff for every $1 \leq i \leq n$ and $1 \leq j \leq n$ for which $|i-j| \geq 2$ there exists an edge E_m of $\mathcal{H}$ such that $\{v_i, v_{i+1}, v_j, v_{j+1}\} = E_m$. ($v_{n+x}$ with $x \geq 0$ denotes the same vertex as v_x.)*

In order to give Dirac type results the definition of the degree must be generalized, too. It is defined now in full generality, however, only some special cases will be used.

Definition 6 *The* degree of a fixed l–tuple *of distinct vertices, $\{v_1, v_2, \ldots, v_l\}$, in a k–uniform hypergraph is the number of edges of the hypergraph containing all $\{v_1, v_2, \ldots, v_l\}$. It is denoted by $d_k(v_1, v_2, \ldots, v_l)$. Furthermore $\delta_k^{(l)}(\mathcal{H})$ denotes minimum of $d_k(v_1, v_2, \ldots, v_l)$ over all l-tuples in $\mathcal{H}$.*

Finally $\lfloor x \rfloor$ denotes the largest integer which is not larger than x and $\lceil x \rceil$ is the smallest integer which is not smaller than x, as usual.

A Dirac type Theorem is proved for k–uniform hypergraphs in Section 2. In the special case $k = 3$ a smaller bound is proved to be sufficient. A natural extremal question is discussed in Section 3: What is the maximum number of edges in a k—uniform hypergraph which does not contain a hamiltonian chain? Section 4 contains a construction which shows that hamiltonian nets differ from hamiltonian chains very much.

2 Hamiltonian Chains in k–uniform Hypergraphs

Theorem 7 *If $\mathcal{H} = (V, \mathcal{E})$ is a k-uniform hypergraph on n vertices with $\delta_k^{(k-1)}(\mathcal{H}) > (1 - \frac{1}{2k})n + 4 - k - \frac{2}{k}$, then $\mathcal{H}$ contains a hamiltonian chain.*

Proof: Let $c = 4 - k - \frac{2}{k}$. Suppose for a contradiction that $\mathcal{H} = (V, \mathcal{E})$ is a k-uniform hypergraph on n vertices that is a counter example to the theorem. Then $\delta_k^{(k-1)}(\mathcal{H}) > (1 - \frac{1}{2k})n + c$, but $\mathcal{H}$ does not contain a hamiltonian chain. Then $k \geq 3$ by Dirac's Theorem. Suppose further that $\mathcal{H}$ is maximal, i. e., if we add any other k–edge to $\mathcal{H}$, then $\mathcal{H}$ will contain a hamiltonian chain. Thus $\mathcal{H}$ contains an open hamiltonian chain $C = (v_1, \ldots, v_n)$.

We will show that there exist different indices p, q, r such that one of the six orderings shown on Fig. 1 is a hamiltonian chain. An ordering is represented as a cycle on the figures. These figures might look complicated at first sight, so we try to explain their structure first. The new ordering or "cycle" will consist from four "paths" of length two (the dashed lines): (v_n, v_{p-1}, v_1), (v_{p-2}, v_r, v_{p+1}), (v_{q-1}, v_p, v_{q+1}) and $(v_{r-1}, v_q, v_{r+1}$. Then these are connected with appropriate parts of C (the solid lines) to complete the new "cycle". If the order of p, q, r is different we get a different "cycle". Each of the six figures corresponds to one of the possible orders of p, q, r.

Lemma 7.1 *For any $(k-1)$–subsets $S = \{s_1, \ldots, s_{k-1}\}$ and $T = \{s_k, \ldots, s_{2k-2}\}$ of V, let*

$$R = \{w \in V | (s_1, \ldots, s_{k-1}, w, s_k, \ldots, s_{2k-2}) \text{ is an open chain}\}.$$

Then $|R| > \frac{n}{2} + k^2 + (c-3)k + 2 =: \alpha$.

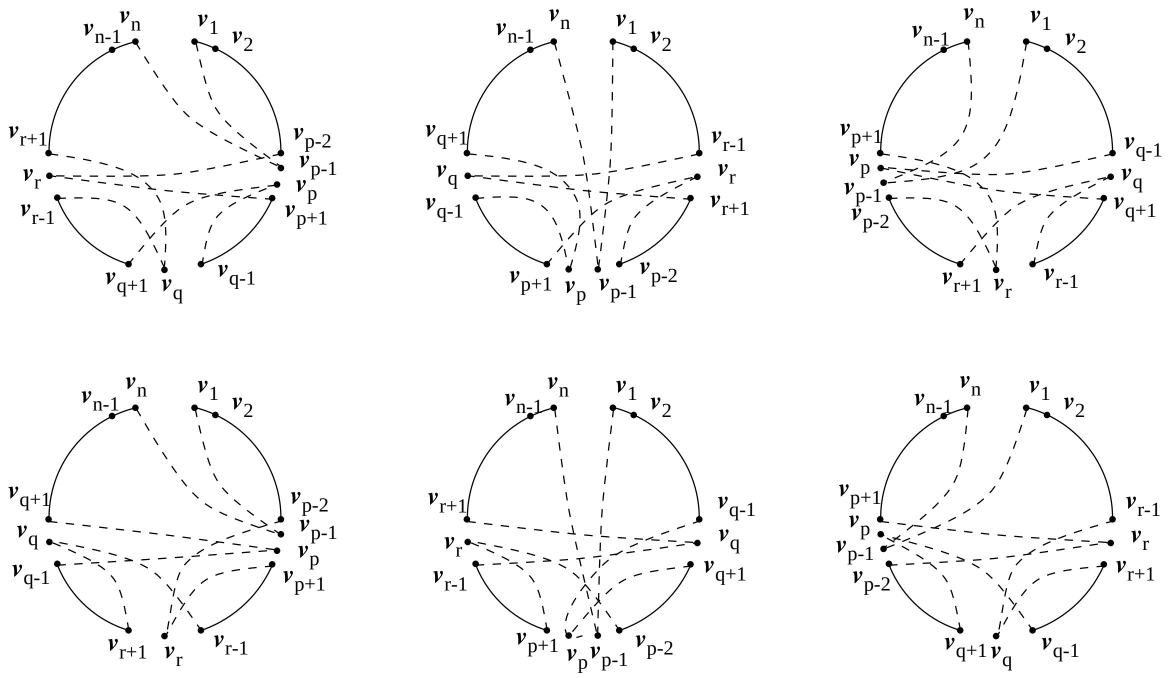

Figure 1: The cycle C.

Proof: Suppose that $w \notin R$. Then there exists $j \in \{1, \ldots, k\}$ such that $\{w\} \cup \{s_j, \ldots, s_{j+k-2}\} \notin \mathcal{E}$. Since $\delta_k^{(k-1)}(\mathcal{H}) > (1 - \frac{1}{2k})n + c$, there are less than $\frac{n}{2k} - c$ such bad w. However, since $\{s_i\} \cup \{s_j, \ldots, s_{j+k-2}\} \notin \mathcal{E}$ if $j \leq i \leq j+k-2$, there are less then $\frac{n}{2k} - c - k + 1$ such bad w in $V - S - T$. Considering all possibilities for j gives that there are less than $\frac{n}{2} - k^2 - (c-1)k$ vertex $w \in V - S - T$ for which $w \notin R$. Since $|V - S - T| = n - 2k + 2$ the lemma follows. ■

Let

$$
\begin{aligned}
R_0^C &= \{w \in V | (v_{n-k+2}, \ldots, v_n, w, v_1, \ldots, v_{k-1}) \text{ is an open chain}\}, \\
R_{1,i}^C &= \{w \in V - \{v_i\} | (v_{i-k+1}, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_{i+k-1}) \text{ is an open chain}\}, \\
R_{2,i}^C &= \{w \in V - \{v_{i-1}, v_i\} | (v_{i-k}, \ldots, v_{i-2}, w, v_{i+1}, \ldots, v_{i+k-1}) \text{ is an open chain}\}.
\end{aligned}
$$

To prove that one of the six orderings shown on Fig. 1 is a hamiltonian chain we have to show that $v_{p-1} \in R_0^C$, $v_p \in R_{1,q}^C$, $v_q \in R_{1,r}^C$ and $v_r \in R_{2,p}^C$.

An auxiliary digraph G is defined to make the proof clearer. Let $X = \{x_{k+1}, \ldots, x_{n-k+1}\}$, $Y = \{y_k, \ldots, y_{n-k+1}\}$ and $Z = \{z_k, \ldots, z_{n-k+1}\}$ (Each x_i, y_i and z_i correspond to v_i.) Let (y_j, x_i) $((z_k, y_j)$, $(x_i, z_k))$ be a directed edge in G iff $v_i \in R_{1,j}^C$ ($v_j \in R_{1,k}^C$, $v_k \in R_{2,i}^C$). (G does not contain any other edges besides these.) Finally let $B = \{x_i \in X | v_{i-1} \notin R_0^C\}$.

Lemma 7.2 *The graph $G - B$ contains a directed triangle, i.e. there exist p, q, r such that x_p, y_q and z_r forms a directed triangle where $x_p \in X - B$.*

Proof: By Lemma 7.1 and the definitions $|R_0^C| > \alpha$, $|R_{1,i}^C| > \alpha - 1$ and $|R_{2,i}^C| > \alpha - 2$, but these sets may contain some vertices from $v_{n-k+2}, \ldots, v_n, v_1, \ldots, v_k$. Thus, by the definition of G, it is clear that

$$
d_G^{\text{out}}(x_i) \quad \geq \quad |R_{2,i}^C| - (2k-2) > \frac{n}{2} + k^2 + (c-5)k + 2 \text{ for every } x_i \in X, \tag{1}
$$

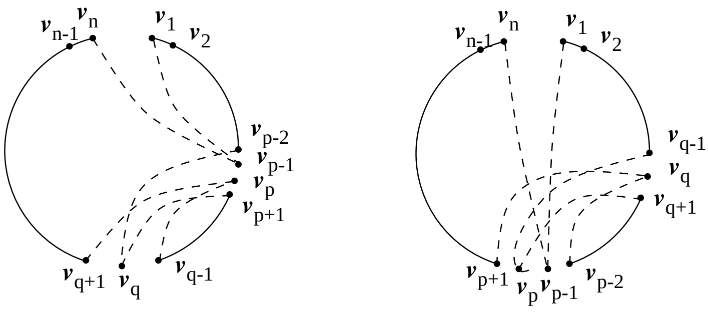

Figure 2: C' and C''

$$d_G^{\text{out}}(y_j) \geq |R_{1,i}^C| - (2k-1) > \frac{n}{2} + k^2 + (c-5)k + 2 \text{ for every } y_j \in Y, \tag{2}$$
$$d_G^{\text{out}}(z_k) \geq |R_{1,i}^C| - (2k-2) > \frac{n}{2} + k^2 + (c-5)k + 3 \text{ for every } z_k \in Z. \tag{3}$$

Obviously $d_G^{\text{out}}(v) = d_{(G-B)}^{\text{out}}(v)$ if $v \in X \cup Z$. On the other hand $d_{(G-B)}^{\text{out}}(y_j) \geq d_G^{\text{out}}(y_j) - |B|$. $x_{n-k+2} \notin X$ and Lemma 7.1 implies that $|X - B| > \alpha - 1$. Thus $|B| < \frac{n}{2} - k^2 - (c-1)k$ and therefore by (2) $d_{(G-B)}^{\text{out}}(y_j) > 2k^2 + (2c-6)k + 2 > 0$ because $c = 4 - k - \frac{2}{k}$. Since the sum of in degrees in Y equals the sum of out degrees in Z it is easy to see from (3) that there exists $y_q \in Y$ for which

$$d_{(G-B)}^{\text{in}}(y_q) > \frac{n}{2} + k^2 + (c-5)k + 3. \tag{4}$$

As we have seen $d_{(G-B)}^{\text{out}}(y_q) \geq 1$, so let $(y_q, x_p) \in E(G)$ where $x_p \in X - B$.

By construction of G $(x_p, z_q) \in E(G)$ implies that $v_q \in R_{2,p}^C$. Since $(y_q, x_p) \in E(G)$ implies that $v_p \in R_{1,q}^C$ and $x_p \in X - B$ implies that $v_{p-1} \in R_0^C$ we obtain that

$$C' = (v_1, v_2, \ldots, v_{p-2}, v_q, v_{p+1}, v_{p+2}, \ldots, v_{q-1}, v_p, v_{q+1}, v_{q+2}, \ldots, v_n)$$

is a hamiltonian chain if $p < q$ or

$$C'' = (v_1, v_2, \ldots, v_{q-1}, v_p, v_{q+1}, v_{q+2}, v_{p-2}, v_q, v_{p+1}, v_{p+2}, \ldots, v_n, v_{p-1})$$

is a hamiltonian chain if $p > q$ (see Fig. 2) which contradicts our assumption.

Thus we may suppose that $(x_p, z_q) \notin E(G)$. By the definition of G $(z_q, y_q) \notin E(G)$. Now (1) and (4) implies that exists $z_r \in Z$ such that $(z_r, y_q) \in E(G)$, $(y_q, z_r) \in E(G)$, $r \neq p$ and $r \neq q$ because

$$\left(\frac{n}{2} + k^2 + (c-5)k + 2\right) + \left(\frac{n}{2} + k^2 + (c-5)k + 3\right) - |Z - \{z_q\}| = 0,$$

since $|Z - \{z_q\}| = n - 2k + 1$ and $c = 4 - k - \frac{2}{k}$. ■

Let x_p, y_q, z_r be a directed triangle in $G - B$ which exists by Lemma 7.2. By the construction of G, $v_p \in R_{1,q}^C$, $v_q \in R_{1,r}^C$ and $v_r \in R_{2,p}^C$ clearly holds and $v_{p-1} \in R_0^C$ is true since $x_p \in X - B$. These imply that one of the six orderings shown on Fig. 1 is a hamiltonian chain. ■

Now try to give some heuristics how we got Fig. 1 and 2. The basic idea was to generalize the proof of Dirac's Theorem which may be sketched as follows. Take a maximal counter example. This contains a hamiltonian path. Then we can prove that it is possible to "cut" the hamiltonian path into two paths such that the generated four endvertices can be connected by two new edges of the graph to obtain a hamiltonian cycle. If $k \geq 3$ then serious restrictions arise about the new edges. Let $I_x(a) = \{v_{x-r}, v_{x-r+1}, \ldots, v_{x+r}\}$ an "interval of vertices on P" the open hamiltonian chain with center v_x and radius r. Suppose that when we

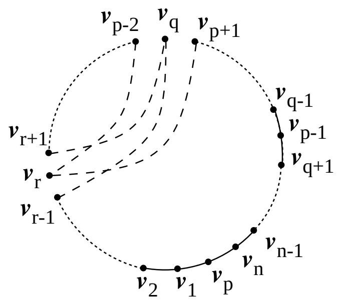

Figure 3: The cycle obtained from C'.

want to prove that C is hamiltonian chain. It must be proved that a new edge E is an edge of the hypergraph where $|E \cap I_x(r)| \geq 2$ and $|E \cap I_y(r)| \geq 2$ (x, y are two independent "variables"). This will not work because even if the degree of all $(k-1)$–element subset is more than $n-$const. it is still possible that there is no such edge E in the hypergraph. This shows that the cycle C cannot contain a part like $\ldots, v_{x-1}, v_x, v_y, v_{y+1}, \ldots$ if $k \geq 4$. (If $k = 3$ then there is no such problem since $|E| = 3$ cannot be partitioned into two part with both parts at least 2.) By similar reasons it cannot contain a part like $\ldots, v_{n-1}, v_n, v_x, v_{x+1} \ldots$ or $\ldots, v_{x-1}, v_x, v_1, v_2 \ldots$ and this also make problems if $k = 3$.

Because of these problems one can easily see that one "cut" is not enough now, we need at least two "cuts". By making two cuts we get 6 endvertices, but there are a lot of obvious restrictions how these can be connected. However, we get a better bound if we use three "cuts" and "connect" the "path" according to Fig. 1.

It is worth mentioning that the proof of Theorem 7 was first proved by a different argument, but later it turned out that it can be reformulated to the idea explained above. On the other hand, the idea of the original proof may be of some interest, too.

In the proof of Dirac's Theorem the endvertices of the hamiltonian path fortunately have some good property. If $k \geq 3$ the open Hamiltonian chain unfortunately does not have the property which is needed to continue the proof similarly to Dirac's proof.

However it can be proved that there exists an open chain of length $n-1$ such that it has the good property.

Lemma 7.3 *If $\mathcal{H} = (V, \mathcal{E})$ is a k-uniform hypergraph on n vertices with $\delta_k^{(k-1)}(\mathcal{H}) > (1 - \frac{1}{2k})n + c$, then $\mathcal{H}$ contains an open chain $C' = (w_1, \ldots, w_{n-1})$ such that $|\{i | w_n \in R_{i,1}^{C'}\}| > \frac{n}{2} + k^2 + (c-7)k + 3$*

Proof: Using the auxiliary graph defined in the proof of Theorem 7 it is easy to see that there exists y_q and x_p such that $(y_q, x_p) \in E(G)$ and $d_G^{\text{in}}(y_q) > \frac{n}{2} + k^2 + (c-5)k + 3$. Thus

$$(v_{p+1}, v_{p+2}, \ldots, v_{q-1}, v_{p-1}, v_{q+1}, v_{q+2}, \ldots, v_n, v_p, v_1, v_2, \ldots, v_{p-2})$$

or

$$v_{p+1}, v_{p+2}, \ldots, v_n, v_p, v_1, v_2, \ldots, v_{q-1}, v_{p-1}, v_{q+1}, v_{q+2}, \ldots, v_{p-2})$$

is an open chain such that $|\{i | v_q \in R_{i,1}^{C'}\}| > \frac{n}{2} + k^2 + (c-7)k + 3$. ■

Now if we are lucky then $v_q \in R_{p,2}^{C'}$ thus the orderings on Fig. 2 are hamiltonian chains. If not then it is easy to show that we can find a "cut" on C' (it will be v_r) such that we can connect the ends of C' through v_r and connect v_{r-1} and v_{r+1} through v_q. (See Fig 3.)

The bound given in Theorem 7 is probably not best possible. However, it is quite easy to show that this bound cannot be lower than $\lfloor \frac{n-k+2}{2} \rfloor$ for k–uniform hypergraphs.

Definition 8 *Let $\mathcal{S}_n^{(k)}$ be a k–uniform hypergraph on the vertex set $V\left(\mathcal{S}_n^{(k)}\right) = X \cup Y$ where $X \cap Y = \emptyset$, $|X| = \lfloor \frac{n}{2} \rfloor$ and $|Y| = \lceil \frac{n}{2} \rceil$. ($X$ and Y are shortly called the* sides *of the hypergraph.) $\mathcal{E}\left(\mathcal{S}_n^{(k)}\right)$ contains every k–element subset E of $X \cup Y$ for which $E \cap X \neq \lfloor \frac{k}{2} \rfloor$.*

Theorem 9 *$\mathcal{S}_n^{(k)}$ does not contain a hamiltonian chain and for every $(k-1)$–element subset $\{p_1, \dots p_{k-1}\}$ of the vertex set*

$$d_k(p_1, \dots, p_{k-1}) \geq \left\lfloor \frac{n-k}{2} \right\rfloor$$

holds.

Proof: Suppose that $(v_1, \dots, v_n)$ is a hamiltonian chain in $\mathcal{S}_n^{(k)}$. Let $E_i = \{v_i, \dots, v_{i+k-1}\}$ and $x_i = |E_i \cap X|$. It is easy to see that $x_{i+1} = x_i$ iff v_i and v_{i+k} are on the same side of the hypergraph and $|x_{i+1} - x_i| = 1$ iff they are in different sides. The equation

$$\sum_{i=1}^{n} x_i = k \left\lfloor \frac{n}{2} \right\rfloor \tag{5}$$

holds because each vertex of X is counted once for each edge containing it, thus exactly k times. Using (5) we conclude that there is an index q such that $x_q \geq \lceil \frac{k}{n} \lfloor \frac{n}{2} \rfloor \rceil$ and an index r such that $x_r \leq \lfloor \frac{k}{n} \lfloor \frac{n}{2} \rfloor \rfloor$. On the other hand, the numbers x_i form a sequence where two consecutive members differ in at most one. Hence every integer in the closed interval $x = \left[\lfloor \frac{k}{n} \lfloor \frac{n}{2} \rfloor \rfloor, \lceil \frac{k}{n} \lfloor \frac{n}{2} \rfloor \rceil \right]$ must occur in the sequence. If n is even then $I = \left[\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil \right]$ and if n is odd then $I = \left[\left\lfloor \frac{k(n-1)}{2n} \right\rfloor, \left\lceil \frac{k(n-1)}{2n} \right\rceil \right]$ thus clearly $\lfloor \frac{k}{2} \rfloor \in I$ since $k < n$. It follows that there is an edge in the hypergraph which contains exactly $\lfloor \frac{k}{2} \rfloor$ vertices of X. But there is no such edge in $\mathcal{S}_n^{(k)}$ by definition, a contradiction.

To prove the degree condition suppose first that $|\{v_1, \dots, v_{k-1}\} \cap X| = \lfloor \frac{k}{2} \rfloor$ and

$$|\{v_1, \dots, v_{k-1}\} \cap Y| = \left\lceil \frac{k}{2} \right\rceil - 1.$$

Then adding any vertex to $\{v_1, \dots, v_{k-1}\}$ from X yields an edge of the hypergraph therefore $d_k(v_1, \dots, v_{k-1}) \geq \lfloor \frac{n}{2} \rfloor - \lfloor \frac{k}{2} \rfloor$ holds in this case.

Suppose now that $|\{v_1, \dots, v_{k-1}\} \cap X| = \lfloor \frac{k}{2} \rfloor - 1$ and $|\{v_1, \dots, v_{k-1}\} \cap Y| = \lceil \frac{k}{2} \rceil$. Then adding any vertex to $\{v_1, \dots, v_{k-1}\}$ from Y yields an edge of the hypergraph therefore $d_k(v_1, \dots, v_{k-1}) \geq \lceil \frac{n}{2} \rceil - \lceil \frac{k}{2} \rceil$ holds in this case.

For any other $(k-1)$–element subset $d_k(v_1, \dots, v_{k-1}) \geq n - k + 1$. Therefore

$$d_k(v_1, \dots, v_{k-1}) \geq \min \left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil \right) = \left\lfloor \frac{n-k}{2} \right\rfloor$$

holds for any $(k-1)$–element subset of the vertex set. ■

3 Many Edges and No Hamiltonian Chain

A natural extremal question about the hamiltonian cycle is that how may edges a graph without a hamiltonian cycle can contain. The extremal case is a complete graph on $(n-1)$ vertices completed by a vertex of degree one. It is trivial to prove that this graph does not contain a hamiltonian cycle, but it is not straightforward to prove that this is the extremal case.

The same question is raised for hypergraph and hamiltonian chain. However, even determining the extremal case looks more difficult.

Theorem 10 *If $\mathcal{H}$ is a k–uniform hypergraph on n vertices satisfying*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{n-1}{n} \binom{n}{k} \tag{6}$$

then $\mathcal{H}$ contains a hamiltonian chain.

Proof: Let m denote the number of missing edges (the k–element subsets which are not edges of $\mathcal{H}$). By (6) we obtain

$$m < \binom{n}{k} - \frac{n-1}{n}\binom{n}{k} = \frac{1}{n}\binom{n}{k}.$$

It is a simple matter to prove that there are $\frac{1}{2}(n-1)!$ different hamiltonian chains in $\mathcal{K}_n^{(k)}$. It is also trivial that $\frac{1}{2}k!(n-k)!$ of these contain a fixed edge. Obviously, if a hamiltonian chain in $\mathcal{K}_n^{(k)}$ contains an edge which is not an edge of $\mathcal{H}$ then it is not a hamiltonian chain in $\mathcal{H}$. Thus the number of hamiltonian chains in $\mathcal{H}$ is

$$\geq \frac{1}{2}(n-1)! - \frac{1}{2}k!(n-k)!m > 0,$$

our claim is proved. ■

At an earlier stage we found a construction with a fairly large number of edges containing no hamiltonian chain. Later better constructions were found but surprisingly this is still the best one in the special case when $k = 3$ and $3|(n-1)$. Therefore this construction may be of some interest in general, too.

Theorem 11 *For any $n \geq 6$ and $3 \leq k \leq n-3$ there exists a k–uniform hypergraph $\mathcal{H}(n,k)$ which does not contain a hamiltonian chain and satisfies*

$$|\mathcal{E}(\mathcal{H}(n,k))| = \binom{n-1}{k^*} + q\binom{2k^*-3}{k^*-1} + \binom{r}{k^*-1} \tag{7}$$

where q and $r < q$ are the unique nonnegative integers satisfying $n-1 = q(2k^*-3)+r$ and

$$k^* = \begin{cases} k & \text{if } k \leq \frac{n}{2} \\ n-k & \text{if } k > \frac{n}{2}. \end{cases}$$

Proof: We first define the hypergraph $\mathcal{H}(n,k)$ only if $k \leq \frac{n}{2}$, thus $k^* = k$ in this part of the proof. Let us partition the vertex set $V - \{v_n\}$ into $q+1$ sets $V_1, \ldots, V_{q+1}$ such that $|V_1| = \ldots = |V_q| = 2k-3$ and $|V_{q+1}| = r$. The partition have at least two nonempty sets since $n-1 > 2k-3$ holds if $k \leq \frac{n}{2}$. The edge set $\mathcal{E}(\mathcal{H}(n,k))$ consists of all k–element subset of $V - \{v_n\}$, denoted by $\mathcal{E}_1(\mathcal{H}(n,k))$, and all $(k-1)$–element subset of V_i $(1 \leq i \leq q+1)$ extended with the vertex v_n, denoted by $\mathcal{E}_2(\mathcal{H}(n,k))$. It follows immediately that (7) holds.

To prove that $\mathcal{H}(n,k)$ does not contain a hamiltonian chain suppose the contrary. Without loss of generality we can assume that this hamiltonian chain is $(v_1, \ldots, v_n)$. The *restriction* of the hamiltonian chain on $V - \{v_n\}$ is a $(k-1)$–uniform open chain of length $2k-3$ connecting v_{n-k+1} and v_{k-1}. On the other hand the restriction of $\mathcal{E}_2(\mathcal{H}(n,k))$ on $V - \{v_n\}$ consists of q copies of $K_{2k-3}^{(k-1)}$ and a $K_r^{(k-1)}$ which clearly cannot contain an open chain of length $2k-3$, a contradiction.

Now the hypergraph $\mathcal{H}(n,k)$ is defined if $k > \frac{n}{2}$. $\mathcal{H}(n,k)$ is obtained from $\mathcal{H}(n,n-k)$ (which is already defined) setting $V(\mathcal{H}(n,k)) = V(\mathcal{H}(n,n-k))$ and replacing each edge $E_i \in \mathcal{H}(n,n-k)$ by the edge $V(\mathcal{H}(n,n-k)) - E_i$. It is clear that (7) is satisfied. On the other hand it follows immediately from the definition of the hamiltonian chain that $\mathcal{H}(n,k)$ contains a hamiltonian chain if and only if $\mathcal{H}(n,n-k)$ does. Since $\mathcal{H}(n,n-k)$ does not contain a hamiltonian chain, the proof is complete. ■

This was improved by a construction of Tuza [7] giving

$$|\mathcal{E}| = \binom{n-1}{k} + (1-o(1))\binom{n-2}{k-2}.$$

This construction led us to the following construction what we conjecture to be the best possible if $k \geq 4$. The construction of Theorem 12 has the same number of edges if $k = 3$ and $3 \not| (n-1)$ but has one more edge if $k = 3$ and $3|(n-1)$.

Theorem 12 *For any $n \geq 4$ and $2 \leq k \leq n-2$ there exists a k–uniform hypergraph $\mathcal{H}(x,y)$ which does not contain a hamiltonian chain and satisfies*

$$|\mathcal{E}(\mathcal{H}(x,y))| = \binom{n-1}{k} + \binom{n-2}{k-2} = \binom{n}{k} - \binom{n-2}{k-1}. \tag{8}$$

Proof: Let x and y be two distinct vertices of the vertex set. $\mathcal{E}(\mathcal{H}(x,y))$ consists of all k element subsets of the vertex set except those which contain x but do not contain y. In other words, if an edge contains x it must contain y, too. It is clear that (8) holds.

To prove that $\mathcal{H}(x,y)$ does not contain a hamiltonian chain suppose the contrary. Without loss of generality we can assume that this hamiltonian chain is $(v_1,\ldots,v_n)$ and that $x = v_1$. Let $y = v_i$. It will be shown that there is an edge of the hamiltonian chain which contains x but does not contain y. If $k+1 \leq i \leq n$ then the edge $(v_1,\ldots v_k)$ is such an edge. $2 \leq i \leq k$ then the edge $(v_{n-k+i},\ldots,v_{i-1})$ has this property proving our claim. ■

4 Hamiltonian Nets

The following Theorem shows that a lower bound cn with $c<1$ for the degrees of triples is not sufficient to force the existence of a hamiltonian net.

Theorem 13 *For any given integer $n \geq 6$ there exists a 4-uniform hypergraph $\mathcal{H}$ on n vertices which does not contain a hamiltonian net and*

$$d_4(x,y,z) \geq \begin{cases} n-9 & \text{if } n \text{ is odd} \\ n-12 & \text{if } n \text{ is even} \end{cases}$$

holds for any x,y,z triple of vertices.

Proof: The construction of $\mathcal{H}$ is given first for odd n. To shorten notation let $r = \frac{n-1}{2}$. It is a well known fact that one can partition the edges of the complete graph $K_n^{(2)}$ into hamiltonian cycles so let $E\left(K_n^{(2)}\right) = H_1 \cup \cdots \cup H_r$ where $H_1,\ldots,H_r$ are the edge sets of edge disjoint hamiltonian cycles. $\mathcal{H}$ is defined on the vertex set of $K_n^{(2)}$. A four element subset of the vertex set is a hyperedge of $\mathcal{H}$ iff these vertices do not span two nonadjacent edges of the same H_i.

Suppose indirectly that $\mathcal{H}$ contains a hamiltonian net. The cyclic ordering can be considered as a hamiltonian cycle C in $K_n^{(2)}$ which has $n = 2r+1$ edges, of course. Since the edge set is partitioned into r hamiltonian cycles there is an H_i which has at least 3 common edges with C and thus there are at least two of them which are nonadjacent. The four end vertices of these two edges does not form a hyperedge of $\mathcal{H}$ by definition, a contradiction.

We prove now that the degree condition holds for $\mathcal{H}$. Let x,y,z be arbitrary distinct vertices of $\mathcal{H}$. An upper bound will be given for the number of those vertices v for which $\{x,y,z,v\} \notin \mathcal{E}(\mathcal{H})$. Each of the three pairs $\{x,y\},\{x,z\}$ and $\{y,z\}$ is an edge of $K_n^{(2)}$ so each of them is contained in exactly one H_i. Without loss of generality we may assume that $\{x,y\} \in H_1$. z is also a vertex of the hamiltonian cycle formed by the edges of H_1. It is clear that $\{x,y,z,v\} \notin \mathcal{E}(\mathcal{H})$ iff v is a neighbor of z on this hamiltonian cycle. If z is not a neighbor of x or y on this hamiltonian cycle then there are two such vertices, otherwise only one. In the same manner one can prove that there may be at most two other choices for v along the hamiltonian cycle containing $\{x,z\}$ and two other along the one containing $\{y,z\}$. Thus there are at most 6 "bad" vertices v out of the $n-3$ possible choices proving that $d_4(x,y,z) \geq n-9$ always holds in this case.

The proof is similar when n is even. Now let $r = \frac{n-2}{2}$. Obviously it is not possible to partition the edge set of $K_n^{(2)}$ into hamiltonian cycles, however, it is also well known that it can be partitioned into r hamiltonian cycles and a perfect matching. We define a new partition by rearranging this. Let $E\left(K_n^{(2)}\right) = H_1' \cup \cdots \cup H_r' \cup \{e\}$ where each H_i' is the union of a hamiltonian cycle and one of the edges from the perfect matching (each H_i' is a hamiltonian cycle with a chord) and $\{e\}$ is the remaining edge of the perfect matching. $H_1' \cup \cdots \cup H_r'$ can also be considered as a partition of $K_n^{(2)} - \{e\}$ into r edge disjoint sets. $\mathcal{H}$ is defined on the vertex set of $K_n^{(2)}$. A four element subset of the vertex set is a hyperedge of $\mathcal{H}$ iff these vertices do not span two nonadjacent edges of the same H_i'.

Suppose indirectly that $\mathcal{H}$ contains a hamiltonian net. The cyclic ordering can be considered as a hamiltonian cycle C in $K_n^{(2)}$ which have $n = 2r+2$ edges in this case. It may contain e but it has $2r+1$

edges different from e. Since the edge set is partitioned into r sets, there is an H_i' which has at least 3 common edges with C. There are at least two of them which are nonadjacent because otherwise they should form a triangle which is impossible, since they are edges of a hamiltonian cycle. The four end vertices of these two edges do not form a hyperedge of $\mathcal{H}$ by definition, a contradiction.

The proof for the degree condition is similar to the previous case, too. The only difference is that if $\{x, y\}$ is an edge of H_1' then it is possible that z has three neighbors if z is an endvertex of the chord. This gives at most 9 "bad" choices of v out of the $n-3$ possibilities. ■

References

[1] J. C. Bermond, A. Germa, M .C. Heydemann and D. Sotteau: *Hypergraph Hamiltoniens*, Problèmes combinatores et théorie des graphes, Orsey (1976) No. 260 pp. 39-43.

[2] Demetrovics J., Katona G. O. H. and Sali A.: Design type problems motivated by database theory, manuscript (1996)

[3] G. A. Dirac: Some theorems of abstract graphs, *Proc. London Math. Soc. (3)* **2** (1952) pp. 69-81.

[4] G. Fan and H. A. Kierstead: The square of path and cycles, *J. Combinatorial Theory*, **B 63** (1995) pp. 55-64.

[5] G. Fan and H. A. Kierstead: Hamiltonian square-paths, *J. Combinatorial Theory*, **B 67** (1996) pp. 167-182.

[6] P. Seymour: *Problem section, "Combinatorics: Proceedings of the British Combinatorial Conference 1973"*, ed. T. P. McDonough and V. C. Mavron, Cambridge Univ. London/New York (1974) pp. 201-202.

[7] Tuza Zs.: personal communication (1996)