Hamiltonian Path Saturated Graphs with Small Size

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Abstract

A graph G is said to be *hamiltonian path saturated* (HPS for short), if G has no hamiltonian path but any addition of a new edge in G creates a hamiltonian path in G. It is known that an HPS graph of order n has size at most $\binom{n-1}{2}$ and, for $n \geq 6$, the only HPS graph of order n and size $\binom{n-1}{2}$ is $K_{n-1} \cup K_1$. Denote by $sat(n, \text{HP})$ the minimum size of an HPS graph of order n. We prove that $sat(n,\text{HP}) \geq \lfloor \frac{3n-1}{2} \rfloor - 2$. Using some properties of Isaacs' snarks we give, for every $n \geq 54$, an HPS graph G_n of order n and size $\lfloor \frac{3n-1}{2} \rfloor$. This proves $sat(n,\text{HP}) \leq \lfloor \frac{3n-1}{2} \rfloor$ for $n \geq 54$. We also consider $m-$path cover saturated graphs and P_m-saturated graphs with small size.

Keywords: graph, hamiltonian path, path cover, saturated graph, extremal graph.
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1 Introduction

We deal with simple graphs without loops and multiple edges. Let m and n be positive integers such that $n \geq m+1$ and let G be a graph of order n. A subgraph F of G is called an *m–path cover (or briefly mPC) of* G if

1. each component of F is a path, (paths of length 0 are allowed)
2. F has at most m components, and
3. $V(F) = V(G)$,

where $V(G)$ denotes the vertex set of the graph G. Clearly, a 1–path cover of G is a hamiltonian path of G.

G is *m–path cover saturated* (mPCS for short), if G has no m–path cover, but connecting any two nonadjacent vertices by a new edge creates an mPC. We shall then write $G \in m$PCS. We write HPS (hamiltonian path saturated) instead of 1PCS.

Graphs with mPC and mPCS graphs were investigated in several papers, see [4], [6], [15], [16]. Skupień [16] gave the maximum size of mPCS graphs of order n and characterized all such graphs. Also, this notion has some connection to Chvátal's toughness conjecture. A graph G is *t—tough* if $|S| \geq t\omega(G{-}S)$ for every subset S of the vertex set $V(G)$ with $\omega(G{-}S) > 1$. Chvátal conjectured ([7]) that there exists a finite constant t_0 such that every t_0—tough graph is hamiltonian. Bauer et al. [2] presented $(\frac{9}{4} - \epsilon)$–tough graphs without a hamiltonian path for arbitrary $\epsilon > 0$, proving that $t_0 \geq \frac{9}{4}$ if it exists. To improve this bound we need to construct a non-hamiltonian graph which has high toughness. Since adding edges to a graph cannot decrease the toughness, it is easy to see that the best constructions must be hamiltonian cycle saturated graphs.

A graph G is said to be *P_m–saturated* if G has no P_m (a path of order m) as a subgraph, but connecting any nonadjacent vertices by a new edge creates a P_m in G.

We are interested in two functions:

$$sat(n, m) = \min\{e(G) : |G| = n \text{ and } G \in m\text{PCS}\}$$

and

$$sat(n, P_m) = \min\{e(G) : |G| = n,\ G \text{ is } P_m - saturated\},$$

where $|G|$ denotes the order, and $e(G)$ the size of the graph G. We shall write $sat(n, \text{HP})$ in place of $sat(n, 1)$.
$Sat(n, m)$ will denote the set of graphs

$$Sat(n, m) = \{G : |G| = n,\ e(G) = sat(n, m) \text{ and } G \in m\text{PCS}\}.$$

We shall write $Sat(n,\text{HP})$ in place of $Sat(n,1)$.
For a graph G and a subset of its vertices S we denote by $G < S >$ the subgraph of G induced by S.

In Sections 3 and 4 of the present paper we prove that for $n \geq 54$ or $n \in \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$

$$\lfloor \frac{3n-1}{2} \rfloor - 2 \leq sat(n, \text{HP}) \leq \lfloor \frac{3n-1}{2} \rfloor$$

We use this result in Section 5 to estimate $sat(n, m)$ and $sat(p, P_q)$ for certain n, m, p and q.
Note that in [5] Bullock, Frick and Singleton constructed an HPS graph on 18 vertices with 30 edges.
In [17] Zelinka gave a construction of a family of graphs $\mathcal{F}_n$ of order n ($n \in \mathbf{N}$), and conjectured that every HPS graph of order n is in $\mathcal{F}_n$. Since $e(G) = O(n^2)$, for every graph $G \in \mathcal{F}_n$, Theorem 3 disproves this conjecture.

2 The lower bound of $sat(n, m)$

In order to determine a lower bound for $sat(n, m)$ we need the following lemmas concerning the degrees of the vertices of an mPCS graph.

Lemma 1 *Let $G \in mPCS$ and let u be a vertex of degree 2 in G. Then the neighbours of u are adjacent.*

Proof: Suppose to the contrary that the neighbours, w and z, of u are not adjacent. Since $G \in m$PCS, $G \cup \{wz\}$ has an mPC, F_m, which contains wz as an edge. Thus F_m cannot contain both uw and uz. Therefore, u must be an end vertex of one of the paths in F_m. So, one of the paths in F_m starts as $(uwz \ldots)$ or $(uzw \ldots)$. Replacing this path by $(wuz \ldots)$ in the first case and $(zuw \ldots)$ in the second case we obtain an mPC in G which does not contain the "new" edge wz, a contradiction. ■

Lemma 2 *Let $G \in mPCS$ and let F_{m+1} be an $(m+1)PC$ of G. Let w, u, z be three consecutive vertices on some path $P^i = (v_{i,1}, v_{i,2}, \ldots, v_{i,j_i})$ in F_{m+1}, with $j_i \geq 5$. Then the following hold:*

1. *If $d(u) = 2$ and w, u, z are internal vertices of P^i, then $d(w) \geq 4$ and $d(z) \geq 4$.*

2. *If $d(u) = 2$ and w is an end-vertex of P^i, then $d(z) \geq 4$. Similarly, if z is an end-vertex of P^i, then $d(w) \geq 4$.*

3. *If w, u, z are internal vertices of P^i and $d(u) \leq 4$, then $d(w) \geq 3$ or $d(z) \geq 3$.*

Proof: We shall prove the first part of the lemma. By Lemma 1 we have $wz \in E(G)$, so $d(w) > 2$ and $d(z) > 2$. Let us suppose that $d(w) = 3$. Let $w' := v_{i,k-2}$. It will be shown that $w'u \in E(G)$ which leads to a contradiction. Since $G \in m$PCS, $G \cup \{w'u\}$ has an mPC, F_m, which contains $w'u$ as an edge. It is obvious that (w) cannot be a trivial path of F_m. Observe that the path $(\ldots zww'u)$ cannot belong to F_m since replacing it by $(\ldots zuww')$ we obtain an mPC in G - a contradiction. Thus F_m must contain uw or uz. In the first case this implies that the corresponding path of F_m contains a segment $\ldots w'uwz \ldots$ or a segment $\ldots w'uw$ (w is a terminal vertex of a path in F_m). Replacing this part by $\ldots w'wuz \ldots$ ($w'wu$ respectively) we obtain an mPC which does not contain the "new" edge $w'u$, a contradiction. In the second case there are two possibilities: $ww'uz \ldots$ and $wzuw' \ldots$, since w must be covered as well. These can be replaced by $w'wuz \ldots$ and $zuww' \ldots$ to obtain the same type of contradiction.
In a similar manner we can prove the second part of the lemma.
Now we prove the last part of the lemma.

Since w and z are internal vertices of a path of F_{m+1}, they have degree at least two. It will be shown that it is not possible, that both degrees are equal to 2. Let $w' =: v_{i,k-2}$ and $z' =: v_{i,k+2}$. Suppose to the contrary that $d(w) = d(z) = 2$. By Lemma 1 we have $w'u \in E(G)$ and $uz' \in E(G)$. Since $G \in m$PCS, $G \cup \{wz\}$ has an mPC, F_m, which contains wz as an edge.

Case 1: wz is a separate path in F_m.

If there is another path P in which u is an internal vertex then $P = (\ldots w'uz' \ldots)$. By replacing P with $P' = (\ldots w'wuzz' \ldots)$ we obtain an $(m-1)$PC of G, a contradiction. If u is an end vertex of a path of F_m then it is easily seen that there is an $(m-1)$PC of $G \cup \{wz\}$.

Case 2: F_m contains wzu.

If (wzu) is a separate path, then replacing this by wuz we obtain an mPC of G which does not contain the "new" edge.

If F_m contains the segment $\ldots wzuz' \ldots$ then replacing it by $\ldots wuzz' \ldots$ we get the same contradiction.

If F_m contains the end segment $wzuw' \ldots$ then replacing it by $zuww' \ldots$ we get the same contradiction. (The case when F_m contains zuw is symmetric.)

Case 3: F_m contains wzz'.

If F_m also contains $z'u$ then it contains either the end segment $wzz'uw' \ldots$ or the end segment $uz'zww' \ldots$. In each case the end segment can be replaced by $z'zuww' \ldots$ to obtain an mPC of G.

If F_m contains $w'u$ then one path in F_m ends with $\ldots w'u$ and another with $\ldots z'zw$ which means there is a path $(\ldots z'zwuw' \ldots)$ in F_m which can be replaced by $(\ldots z'zuww' \ldots)$ and we get the same type of contradiction.

If u is a trivial path of F_m then this path and the edge wz in F_m may be replaced with $\ldots wuz \ldots$ and we obtain an mPC in G, a contradiction. (The case when F_m contains zww' is symmetric.) ■

Lemma 3 *Let $G \in mPCS$ and let F_{m+1} be an $(m+1)PC$ of G. If w, u, z are three consecutive internal vertices of some path in F_{m+1}, then $d(w) + d(u) + d(z) \geq 9$.*

Proof: Since w, u, z are all internal vertices on some path of F_{m+1} all of them have degree at least 2. By Lemma 1 if two of them have their degree equal to 2 then $d(w) = d(z) = 2$. On the other hand, in this case $d(u) \geq 5$ by Lemma 2. Hence, we may suppose that exactly one of vertices $u,\ w\ z$ has its degree equal to 2. If $d(w) = 2$ then $d(u) \geq 4$ by Lemma 2 and Lemma 3 holds. If $d(z) = 2$ then we obtain the same result by symmetry.

If $d(u) = 2$ then by Lemma 2 $d(w) \geq 4$ and $d(z) \geq 4$ proving our claim. ■

Theorem 1 *Let n and m be positive integers, $n \geq m + 1$. Then*

$$sat(n, m) \geq \frac{3}{2}n - 3(m + 1).$$

.

Proof: Let $G \in Sat(n, m)$. By definition G has no mPC, however, it is easy to see that it has an $(m + 1)$PC. Let $F_{m+1} = \bigcup_{i=1}^{m+1} P^i$ be an $(m + 1)$PC of G, with $P^i = (v_{i,1}v_{i,2} \dots v_{i,j_i})$; $i = 1, \dots, m + 1$. To prove the theorem we estimate the sum of the degrees in G using Lemma 3. Let $P^1, \dots, P^{m_1}$ be the paths with exactly one vertex; $P^{m_1+1}, \dots, P^{m_1+m_2}$ the paths with exactly two vertices; $P^{m_1+m_2+1}, \dots, P^{m_1+m_2+m_3}$ the paths with exactly three vertices and $P^{m_1+m_2+m_3+1}, \dots, P^{m+1}$, the paths with at least four vertices. Denote by $n_l, l = 1, 2, 3$ the number of vertices of G covered by paths of order l. Denote by n_4 the number of vertices of G covered by paths of order at least four, and by m_4 the number of paths with at least four vertices. Clearly n, m, m_1, m_2, m_3, m_4 satisfy $n \geq m + 1$, $m_1 + m_2 + m_3 + m_4 = m + 1$. We have $n_l = lm_l$ for $l = 1, 2, 3$. Thus $n = m_1 + 2m_2 + 3m_3 + n_4$. For the degrees of vertices $v_{i,2}$ and v_{i,j_i-1} we will use the trivial lower bound 2 and the degrees of $v_{i,1}$ and v_{i,j_i} the lower bound 1. (This part of proof could be improved by similar methods, but the proof would be too large compared to the small improvement.)

$$6e(G) = 3 \sum_{v \in V(G)} d(v) = \sum_{i=m_1+m_2+m_3+1}^{m+1} \sum_{k=3}^{j_i-2} (d(v_{i,k-1}) + d(v_{i,k}) + d(v_{i,k+1})) +$$

$$+ \sum_{i=m_1+m_2+m_3+1}^{m+1} [3\,(d(v_{i,1}) + d(v_{i,j_i})) + 2\,(d(v_{i,2}) + d(v_{i,j_i-1})) +$$

$$+ (d(v_{i,3}) + d(v_{i,j_i-2}))] + 3 \sum_{i=m_1+m_2+1}^{m_1+m_2+m_3} (d(v_{i,1}) + d(v_{i,2}) + d(v_{i,3})) +$$

$$+3 \sum_{i=m_1+1}^{m_1+m_2} (d(v_{i,1}) + d(v_{i,2})) + 3 \sum_{i=1}^{m_1} d(v_{i,1}) \geq$$

$$9(n_4-4m_4)+6m_4+8m_4+4m_4+18m_3+6m_2 = 9n_4-18m_4+18m_3+6m_2 =$$
$$= 9n-9m_1-12m_2-9m_3-18m_4 \geq 9n-18m_1-18m_2-18m_3-18m_4 = 9n-18(m+1)$$

The above inequality implies that $e(G) \geq \frac{3}{2}n - 3(m+1)$. ■

3 The lower bound of $sat(n, HP)$

Theorem 2 *Let $G \in Sat(n, HP), n \geq 14$. Then $e(G) \geq \lfloor \frac{3n-1}{2} \rfloor - 2$.*

Proof: To prove the theorem we estimate the sum of degrees using Lemmas 1 and 3. Note that it is sufficient to prove that $6e(G) \geq 9n - 20$. It is clear that the vertices of G may be covered by two vertex disjoint paths P^1, P^2. We shall consider five cases.
Case 1. $|P^1| \geq 5, |P^2| \geq 5, P^1 = (a_1, a_2, \ldots, a_k), P^2 = (b_1, b_2, \ldots, b_{n-k})$.
By Lemma 3, we have

$$\begin{aligned}
6e(G) = 3 \sum_{v \in V(G)} d(v) = \\
&= \sum_{i=3}^{k-2} (d(a_{i-1}) + d(a_i) + d(a_{i+1})) + \\
&3\,(d(a_1) + d(a_k)) + 2\,(d(a_2) + d(a_{k-1})) + d(a_3) + d(a_{k-2}) + \\
&\sum_{i=3}^{n-k-2} (d(b_{i-1}) + d(b_i) + d(b_{i+1})) + \\
&3\,(d(b_1) + d(b_{n-k})) + 2\,(d(b_2) + d(b_{n-k-1})) + d(b_3) + d(b_{n-k-2}) \\
&\geq 9(n-8) + \\
&2(d(a_1) + d(a_2) + d(a_k) + d(a_{k-1}) + d(b_1) + d(b_2) + d(b_{n-k}) + d(b_{n-k-1})) \\
&+ d(a_1) + d(a_3) + d(a_k) + d(a_{k-2}) + d(b_1) + d(b_3) + d(b_{n-k}) + d(b_{n-k-2})
\end{aligned}$$

Since $d(a_i), d(b_j) \geq 1$, for $i = 1, k$; $j = 1, n-k$ and $d(a_i), d(b_j) \geq 2$, for $i = 2, 3, k-1, k-2$, $j = 2, 3, n-k-1, n-k-2$, we have $6e(G) \geq 9n - 36$. Observe that if $d(a_1) + d(a_3) = 3$ then $d(a_1) = 1, d(a_3) = 2$ and the addition of any edge a_2x does not create any hamiltonian path in $G \cup \{a_2x\}$ so $d(a_2) = n-1$ and $6e(G) \geq 9n - 36 + 2(n-3) \geq 11n - 42 \geq 9n - 20$ for $n \geq 11$. Hence we may suppose that $d(a_1) + d(a_3) \geq 4$ and similarly
$d(a_k) + d(a_{k-2}) \geq 4$,
$d(b_1) + d(b_3) \geq 4$,
$d(b_{n-k}) + d(b_{n-k-2}) \geq 4$.
It follows easily from Lemma 1 that $d(a_1) + d(a_2) \geq 4$. If, however, $d(a_1) + d(a_2) = 4$ then, clearly, for any nonadjacent pair of vertices x, y such that $x, y \notin \{a_1, a_2\}$ any hamiltonian path in $G \cup \{xy\}$ starts either from a_1 or a_2. The same situation

is for the couples $\{a_k, a_{k-1}\}$, $\{b_1, b_2\}, \{b_{n-k}, b_{n-k-1}\}$. One may check that for at most two of these four couples we may have the degrees sum equal to 4. Hence $d(a_1)+d(a_k)+d(a_2)+d(a_{k-1})+d(b_1)+d(b_{n-k})+d(b_2)+d(b_{n-k-1}) \geq 18$.
We conclude that $6e(G) \geq 9n-20$.
Case 2. $|P^1| = n-1, |P^2| = 1, P^1 = (a_1, a_2, \ldots, a_{n-1}), P^2 = (a_n)$.

We have
$6e(G) = 3\sum_{v \in V(G)} d(v) = s_1+s_2+s_3+s_4+s_5$ where

$$
\begin{aligned}
s_1 &= \sum_{i=3}^{n-3} (d(a_{i-1}) + d(a_i) + d(a_{i+1})) \\
s_2 &= 3\,(d(a_1) + d(a_{n-1})) \\
s_3 &= 2\,(d(a_2) + d(a_{n-2})) \\
s_4 &= (d(a_3) + d(a_{n-3})) \\
s_5 &= 3d(a_n)
\end{aligned}
$$

If $s_5 = 0$ then, clearly, $G - \{a_n\} = K_{n-1}$ and the theorem holds. Thus we may suppose $s_5 \geq 3$.
If $d(a_1) = d(a_{n-1}) = d(a_n) = 1$ then $G - \{a_1, a_{n-1}, a_n\} = K_{n-3}$, $e(G) = \binom{n-3}{2} + 3 \geq \frac{9n-19}{6}$ and the theorem follows. So we may assume $d(a_1)+d(a_{n-1})+d(a_n) \geq 4$. Observe that if one of the vertices of a_1 or a_{n-1} has its degree equal to 1 then, by Lemma 1, its neighbour on P^1 has degree at least 3. So we may assume that either $d(a_1) \geq 2$ or $d(a_2) \geq 3$. Considering all the possibilities one may check that $s_1+s_2+s_3+s_4+s_5 \geq \frac{9n-19}{6}$.
Case 3. $|P^1| = n-2, |P^2| = 2, P^1 = (a_1, a_2, \ldots, a_{n-2}), P^2 = (a_{n-1}, a_n)$.
Clearly, neither a_{n-1} nor a_n is adjacent to $\{a_1, a_{n-2}\}$. If a_{n-1} or a_n is adjacent to one of the vertices $a_2, \ldots, a_{n-3}$ then we have Case 1 or Case 2. Hence $G = K_2 \cup K_{n-2}$ and $e(G) = \binom{n-2}{2} + 1 \geq \frac{9n-19}{6}$ for $n \geq 14$.
Case 4. $|P^1| = n-3, |P^2| = 3, P^1 = (a_1, a_2, \ldots, a_{n-3}), P^2 = (a_{n-2}, a_{n-1}, a_n)$.
Neither a_{n-2} nor a_n is adjacent to $\{a_1, a_{n-3}\}$. If a_{n-2} or a_n is adjacent to one of the vertices $a_2, \ldots, a_{n-4}$ then we have Case 1 or 2. If $a_{n-2}a_n \in E$, then we have either Cases 1 or 2 or $G = K_3 \cup K_{n-3}$ and $e(G) = \binom{n-3}{2} + 3 \geq \frac{9n-19}{6}$ for $n \geq 14$. Hence we may suppose $d(a_{n-2}) = d(a_n) = 1$. Then the graph $G < a_1, \ldots, a_{n-3}, a_{n-1} >= K_{n-2}$ and the case follows.
Case 5. $|P^1| = n-4, |P^2| = 4, P^1 = (a_1, a_2, \ldots, a_{n-4})$,
$P^2 = (a_{n-3}, a_{n-2}, a_{n-1}, a_n)$.
Clearly, neither a_{n-3} nor a_n is adjacent to $\{a_1, a_{n-4}\}$. If a_{n-3} or a_n is adjacent to one of the vertices $a_2, \ldots, a_{n-5}$ then we have Case 1 since we are assuming $n \geq 14$.
If $a_{n-3}a_n \in E$ then for similar reasons none of the vertices
$a_{n-3}, a_{n-2}, a_{n-1}, a_n$ is adjacent to any vertex of the set $\{a_1, \ldots, a_{n-4}\}$ and thus

$G = K_4 \cup K_{n-4}$ and the proof follows. So we may suppose $a_{n-3}a_n \notin E$. We may also assume that one of the vertices a_{n-2} or a_{n-1} is adjacent to the path P^1, say $a_{n-1}a_{i_0} \in E$ with $i_0 \in \{1, \dots, n-4\}$.
If $a_{n-2}a_n \in E$ then we have the path $(a_{n-1}, a_n, a_{n-2}, a_{n-3})$ with $a_{n-1}a_{i_0} \in E$ and it is easy to see that we have Case 1. So we assume $a_{n-2}a_n \notin E$.
If $a_{n-3}a_{n-1} \in E$ then we have the path $(a_{n-2}, a_{n-3}, a_{n-1}, a_n)$ and the rest of the proof runs as before, with a_{n-2} and a_{n-3} interchanged. So we may assume $a_{n-3}a_{n-1} \notin E$ and thus $d(a_n) = d(a_{n-3}) = 1$. The graph $G' = G \cup \{a_{n-2}a_n\}$ has the hamiltonian path $P = (a_{n-3}, a_{n-2}, a_n, a_{n-1}, \dots x)$. The paths $(a_{n-3}, a_{n-2}, a_{n-1}, \dots x), (a_n)$ cover the vertices of G as in Case 2, which completes the proof. ■

4 HPS graphs with small size

The following lemma is a consequence of some results (theorems 5, 7, 9 and 11) given in [9] by L.H. Clark, R.C. Entringer and H.D. Shapiro (see also [8]).

Lemma 4 *For every n even, $n \geq 52$ or $n \in \{20, 28, 36, 38, 40, 44, 46, 48\}$ there is a non hamiltonian graph G_n of order n with the following properties:*

1. *G_n is cubic,*
2. *there is an edge $e = xy$ in G_n such that $N(x) \cap N(y) = \emptyset$ and, for every pair of nonadjacent vertices u, v, the graph $G_n \cup \{uv\}$ has a hamiltonian cycle containing e.* ■

More precisely Clark, Entringer and Shapiro proved that for k odd and sufficiently big, the Isaacs' graphs J_k of order $4k$ defined in [10] and their modifications given in [9] have the property of the thesis of Lemma 4. Their results imply in particular, that for $n \geq 52$ the minimum size of a maximally non-hamiltonian graph of order n is $\lceil \frac{3n}{2} \rceil$. Lin, Jiang, Zhang and Yang in [11] set the values of the size of smallest maximally non-hamiltonian graphs for all remaining orders n. Note that Kalinowski and Skupień proved in [12], [13] that the Isaacs' graphs are maximally non-hamiltonian-connected with minimum size.

Theorem 3 *For every $n \geq 54$ or $n \in \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$ there is an HPS graph of order n and size $\lfloor \frac{3n-1}{2} \rfloor$.*

Proof: Theorem 3 follows from Lemma 4 and the three lemmas given below.

Lemma 5 *Let $G = (V; E)$ be a graph of order n which satisfies the conditions of Lemma 4. Then the graph $G - x$ is hamiltonian.*

Proof: As in the proof of hypohamiltonicity of J_k given in [8] we take a path (xyz). Since $xz \notin E(G)$, the edge xy is contained in hamiltonian path $P = (xy, \ldots, z)$ of G. Deleting from P the vertex x and adding the edge yz we obtain a hamiltonian cycle of $G - x$. ■

Lemma 6 *Let $G = (V; E)$ be a graph satisfying the conditions of Lemma 4. Then the graph $G' = (V'; E')$ where $V' = V \cup \{z_1, z_2\}$, $E' = E \cup \{xz_1, yz_2\}$ is HPS.*

Proof: Since G is not hamiltonian, it is clear that G' has no hamiltonian path. Let uv be a new edge of G'. We shall prove that $G' \cup \{uv\}$ has a hamiltonian path. We shall consider 4 cases.
Case 1. $u = z_1, v = z_2$
Let P be a hamiltonian path of G starting from x. Then z_2z_1P is a hamiltonian path of $G' \cup \{uv\}$.
Case 2. $u, v \notin \{z_1, z_2\}$
The graph $G \cup \{uv\}$ contains a hamiltonian cycle through xy. A hamiltonian path of $G' \cup \{uv\}$ is easy to obtain.
Case 3. $u = z_1, v = y$
Let w be a vertex of G such that $wy \notin E$. Let $P = (y, x, v_1, \ldots, v_{n-3}, w)$ be a hamiltonian path in G through xy. $P' = (z_2, y, z_1, x, v_1, \ldots, v_{n-3}, w)$ is a hamiltonian path of $G' \cup \{uv\}$.
Case 4. $u = z_1, v \in V - \{x, y\}$
If $yv \notin E$ then consider a hamiltonian path $P = (v, \ldots, y)$, then z_1Pz_2 is a hamiltonian path of $G' \cup \{uv\}$. So we may suppose $yv \in E$. By Lemma 5 the graph $G - \{x\}$ is hamiltonian and since in $G - \{x\}$ the degree of y is equal to 2, there is in $G - \{x\}$ a hamiltonian path P from y to v. Then z_2Pz_1x is a hamiltonian path of $G' \cup \{uv\}$ ■

The proof of the following result is similar to that of Lemma 6.

Lemma 7 *Let $G = (V; E)$ be a graph of order n which satisfies the assumption of Lemma 4. Then the graph $G' = (V'; E')$ where $V' = V \cup \{z_1, z_2, z_3\}$, $E' = E \cup \{xz_1, yz_2, yz_3, z_2z_3\}$ is HPS.*

■

5 P_q- saturated graphs and $m-$ path cover saturated graphs

In [14] Kászonyi and Tuza gave the minimum size of a P_q–saturated graph G for q sufficiently small with respect to the order of G by proving the following theorem.

Theorem 4 (L. Kászonyi and Zs. Tuza)
Let
$$a_q = \begin{cases} 3 \cdot 2^{k-1} - 2 & \textit{if} \quad q = 2k, k > 2, \\ 2^{k+1} - 2 & \textit{if} \quad q = 2k+1, k \geq 2, \end{cases} .$$
Then, for $n \geq a_q$,
$$sat(n, P_q) = n - \lfloor \tfrac{n}{a_q} \rfloor .$$

■

Corollary 5 *Let $p \geq q$.*

1. *For q even, $q - 2 \in \{20, 28, 36, 38, 40, 44, 46, 48\}$ or $q - 2 \geq 52$*
$$sat(p, P_q) \leq p + \frac{q}{2} - 1$$

2. *For q odd, $q - 3 \in \{20, 28, 36, 38, 40, 44, 46, 48\}$ or $q - 3 \geq 52$*
$$sat(p, P_q) \leq \frac{3p}{2}$$

Proof: Let $G = (V; E)$ be a graph of order n satisfying the assumptions of Lemma 4. We shall construct three families of graphs.

1. $H^1_{n+l+1} = (V^1; E^1)$ where $V^1 = V \cup \{v_0, v_1, \ldots, v_l\}$, $l \geq 1$, $E^1 = E \cup \{xv_0, yv_1, yv_2, \ldots, yv_l\}$,
2. $H^2_{n+2l+1} = (V^2; E^2)$ where $V^2 = V \cup \{v_0, v_1, \ldots, v_l, w_1, \ldots, w_l\}$, $l \geq 1$, $E^2 = E \cup \{v_0x, v_iw_i, yv_i, yw_i; i = 1, \ldots, l\}$
3. $H^3_{n+2l+2} = (V^3; E^3)$ where $V^3 = V \cup \{v_0, v_1, \ldots, v_l, w_0, w_1, \ldots, w_l, \}$, $l \geq 1$, $E^3 = E \cup \{v_iw_i; i = 0, \ldots, l, v_0x, w_0x, v_iy, w_iy; i = 1, \ldots, l\}$.

The corollary follows from the following three observations.

1. the graph H^1_{n+l+1} is P_{n+2}–saturated of order $p = n+l+1$ and size $\frac{3n}{2}+l+1$,
2. the graph H^2_{n+2l+1} is P_{n+3}–saturated of order $p = n + 2l + 1$ and size $\frac{3n}{2} + 3l + 1$,
3. the graph H^3_{n+2l+2} is P_{n+3}–saturated of order $p = n + 2l + 2$ and size $\frac{3n}{2} + 3(l + 1)$ (note that H^3_{n+2l+2} is also P_{n+4}–saturated).

. ■

Observe that if $G = (V; E)$ is an HPS graph without a total vertex then the graph $G' = (V'; E')$ with $G' = G \cup (m-2)K_2 \cup K_1$ is mPCS. By Theorem 3 we have the following corollary.

Corollary 6 *For* $m \geq 2, n-2m+3 \in \{22,23,30,31,38,39,40,41,42,43,46,47,48,49,50,51\}$ *or* $n-2m+3 \geq 54$

$$sat(n,m) \leq \frac{3n}{2} - 2m + 2$$

■

Observe that the values of $sat(n,\text{HP})$, $sat(p, P_q)$ and $sat(n,m)$ remain undetermined.

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