

# Extremal $k$ -edge-hamiltonian Hypergraphs

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Keywords:  $k$ -edge-hamiltonian, hamiltonian cycle, hypergraph

## Abstract

An  $r$ -uniform hypergraph is  $k$ -edge-hamiltonian iff it still contains a hamiltonian chain after deleting any  $k$  edges of the hypergraph. What is the minimum number of edges in such a hypergraph? We give lower and upper bounds for this question for several values of  $r$  and  $k$ .

## 1 Introduction

Let  $\mathcal{H}$  be a  $r$ -uniform hypergraph on the vertex set  $V(\mathcal{H}) = \{v_1, v_2, \dots, v_n\}$  where  $n > r$ . For simplicity of notation  $v_{n+x}$  with  $x \geq 0$  denotes the same vertex as  $v_x$  (unless stated otherwise). The set of the edges,  $r$ -element subsets of  $V$ , is denoted by  $\mathcal{E}(\mathcal{H}) = \{E_1, E_2, \dots, E_m\}$ . We will write simply  $V$  for  $V(\mathcal{H})$  and  $\mathcal{E}$  for  $\mathcal{E}(\mathcal{H})$  if no confusion can arise.

In [1] the authors defined the notion of a hamiltonian-chain.

**Definition 1** *A cyclic ordering  $(v_1, v_2, \dots, v_n)$  of the vertex set is called a hamiltonian chain iff for each  $1 \leq i \leq n$   $\{v_i, v_{i+1}, \dots, v_{i+r-1}\} =: E_j$  is an edge of  $\mathcal{H}$ . An ordering  $(v_1, v_2, \dots, v_{l+1})$  of a subset of the vertex set is called an open chain of length  $l$  between  $v_1$  and  $v_{l+1}$  iff for each  $1 \leq i \leq l - r + 2$  there exists an edge  $E_j$  of  $\mathcal{H}$  such that  $\{v_i, v_{i+1}, \dots, v_{i+r-1}\} = E_j$ . An open chain of length  $n - 1$  is an open hamiltonian chain. A cyclic ordering  $(v_1, v_2, \dots, v_l)$  of a subset of the vertex set is called a chain of length  $l$  iff for every  $1 \leq i \leq l$  there exists an edge  $E_j$  of  $\mathcal{H}$  such that  $\{v_i, v_{i+1}, \dots, v_{i+r-1}\} = E_j$ . (Now  $v_{l+x}$  denotes the same vertex as  $v_x$ ).*

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\*Research supported by OTKA grants OTKA T 029772 and T 030059, and The Japan Society for the Promotion of Science

**Definition 2** *A hypergraph is hamiltonian if it contains a hamiltonian-chain and it is  $k$ -edge-hamiltonian if by the removal of any  $k$  edges a hamiltonian hypergraph is obtained.*

The notion of the degree is also extended, it is defined below in full generality, however, only some special cases will be used.

**Definition 3** *The degree of a fixed  $l$ -tuple of distinct vertices,  $\{v_1, v_2, \dots, v_l\}$ , in a  $r$ -uniform hypergraph is the number of edges of the hypergraph containing the set  $\{v_1, v_2, \dots, v_l\}$ . It is denoted by  $d_r(v_1, v_2, \dots, v_l)$ . Furthermore  $\delta_r^{(l)}(\mathcal{H})$  denotes the minimum of  $d_r(v_1, v_2, \dots, v_l)$  over all  $l$ -tuples of vertices in  $\mathcal{H}$ . The neighborhood of a vertex  $v$  is defined by*

$$N_{\mathcal{H}}(v) := \{E - \{v\} \mid v \in E, E \in \mathcal{E}(\mathcal{H})\}.$$

The main aim of the present article is to investigate minimum size  $k$ -edge-hamiltonian hypergraphs. In [2, 3] the authors settle this question for graphs.

**Theorem 4** [2, 3] *The number of edges in a minimum  $k$ -edge-hamiltonian graph on  $n \geq k + 3$  vertices is  $\lceil n(k + 2)/2 \rceil$ .*

Since the degree of any vertex in a  $r$ -uniform hamiltonian chain is  $r$ , the minimum degree in a  $k$ -edge-hamiltonian hypergraph is at least  $r + k$ , so the number of edges is at least  $\lceil n(r + k)/r \rceil$ . For  $r = 2$  this shows that the constructions in the above theorem are best possible. However, for  $r > 2$  this lower bound is not best possible.

## 2 3-uniform hypergraphs

If a hypergraph contains  $k + 1$  edge-disjoint hamiltonian chains, then it is clearly  $k$ -edge-hamiltonian. This observation leads to the trivial upper bound on the minimum number of edges:  $(k + 1)n$ . If  $k = 1$  then the following slightly better upper bound is obtained.

**Theorem 5** *There exists a 1-edge-hamiltonian 3-uniform hypergraph  $\mathcal{H}$  on  $n$  vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{11}{6}n + o(n).$$

PROOF: Let  $\mathcal{V}(\mathcal{H}) := \{w_1, \dots, w_p, v_1, \dots, v_q\}$  where  $p = \lceil n/6 \rceil$  and  $q = n - p$ . There are two types of edges in  $\mathcal{H}$ . The first kind of edges form a chain on  $\{v_1, \dots, v_q\}$ ,

$$\mathcal{E}_1(\mathcal{H}) := \left\{ \{v_i, v_{i+1}, v_{i+2}\} \mid 1 \leq i \leq q \right\}.$$

The second kind connects the rest of the vertices to this chain:

$$\mathcal{E}_2(\mathcal{H}) := \left\{ \{w_i, v_{5(i-1)+j}, v_{5(i-1)+j+1}\} \mid 1 \leq i \leq p, 1 \leq j \leq 6 \right\}.$$

This means that the neighborhood of  $w_i$  is an ordinary graph, a path of length 6 formed by vertices  $v_{5(i-1)+1}, \dots, v_{5(i-1)+7}$ . The neighborhood of  $w_{i+1}$  is also a path of length 6, which begins at  $v_{5(i-1)+6}$ , so  $v_{5(i-1)+6}, v_{5(i-1)+7} \in N(w_i) \cap N(w_{i+1})$  (except maybe for  $N(w_1)$  and  $N(w_p)$  where the overlap is larger if  $6 \nmid n$ ). Let  $\mathcal{E}(\mathcal{H}) := \mathcal{E}_1(\mathcal{H}) \cup \mathcal{E}_2(\mathcal{H})$ , then it is clear that  $|\mathcal{E}(\mathcal{H})| = q + 6p = n + 5\lceil n/6 \rceil = 11n/6 + o(n)$ . (See Fig. 1.)

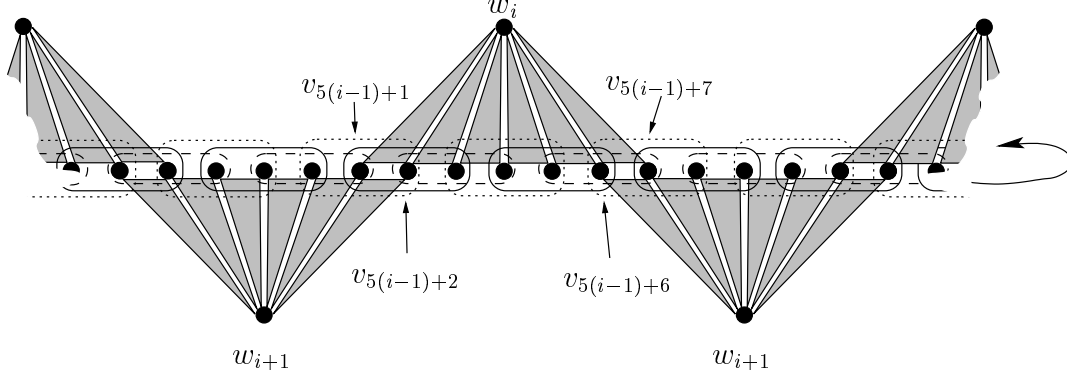


Figure 1: 3-uniform 1-edge-hamiltonian hypergraph

This hypergraph contains many hamiltonian-chains which can be obtained in the following way. Start with the chain formed by  $\{v_1, \dots, v_q\}$  and extend this cycle by inserting the rest of the vertices one by one. It is obvious that we can insert  $w_i$  between any two consecutive vertices of  $v_{5(i-1)+2}, v_{5(i-1)+3}, v_{5(i-1)+4}, v_{5(i-1)+5}, v_{5(i-1)+6}$  (but we cannot insert it between  $v_{5(i-1)+1}$  and  $v_{5(i-1)+2}$  or  $v_{5(i-1)+6}$  and  $v_{5(i-1)+7}$ ). Note that the new chain contains 3 “consecutive edges” of  $N(w_i)$  but it does not contain 2 “consecutive edges” from the original chain (those which contain both neighbors of  $w_i$  in the new chain (See Fig. 2).

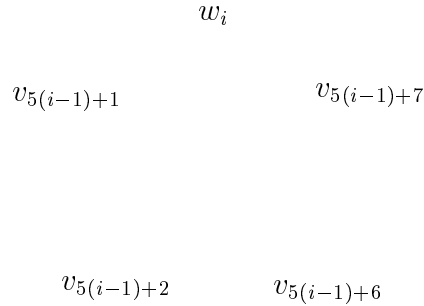


Figure 2: How to insert  $w_i$ ?

Now we prove that  $\mathcal{H}$  is 1-edge-hamiltonian, that is,  $\mathcal{H} - E$  contains a hamiltonian-chain for any  $E \in \mathcal{E}(\mathcal{H})$ .

Suppose that  $E = \{v_t, v_{t+1}, v_{t+2}\} \in \mathcal{E}_1(\mathcal{H})$ . Then it is easy to check that there is a  $w_i$  which we can insert either between  $v_t$  and  $v_{t+1}$  or  $v_{t+1}$  and  $v_{t+2}$ , so the new chain does not contain  $E$  any more. Further, we can insert all other  $w$  vertices into suitable places, hence we obtain the desired hamiltonian-chain (see Fig. 3), for example the following one

$$v_t, v_{t+1}, w_i, v_{t+2}, v_{t+3}, \dots, v_{t+5}, v_{t+6}, w_{i+1}, v_{t+7}, v_{t+8}, \dots, v_{t+5j}, v_{t+5j+1}, w_{i+j}, v_{t+5j+2}, \dots$$

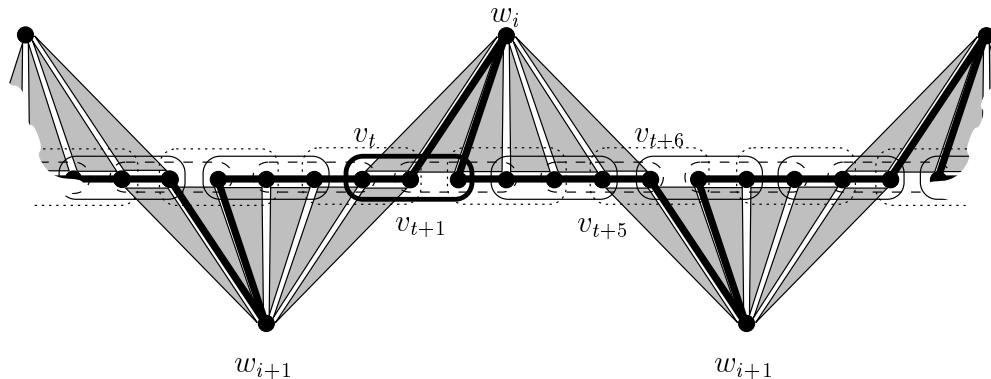


Figure 3: Hamiltonian chain in  $\mathcal{H} - E$

On the other hand, if  $w_i \in E$  for some  $i$  then it is clear that  $N(w_i) - E$  always contains 3 “consecutive edges”, therefore  $w_i$  can be inserted into the chain formed by  $\{v_1, \dots, v_q\}$ . Inserting the rest of the vertices in the same way as in the other case, we obtain a hamiltonian-chain of  $\mathcal{H} - E$ .  $\square$

**Theorem 6** *For any 1-edge-hamiltonian 3-uniform hypergraph  $\mathcal{H}$  on  $n \geq 5$  vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{14}{9}n$$

*holds.*

PROOF: Observe that the neighborhood of a vertex in a hamiltonian-chain is a path on 4 distinct vertices, a  $P_4$ . Let us call a graph *stable* if it contains a  $P_4$  after deleting any edge of the graph. Thus, the neighborhood of every vertex of a 1-edge-hamiltonian graph is stable. We also call a vertex of the hypergraph *stable* iff its neighborhood is stable.

It is easy to check that the only stable graph with 4 edges is the  $C_4$ , the cycle with 4 edges. All other stable graphs contain at least 5 edges. In fact, there are only 3 stable graphs with 5 edges (see Fig. 4).

Let  $\mathcal{H}$  be a 1-edge-hamiltonian 3-uniform hypergraph and let  $v_1, \dots, v_n$  be a hamiltonian chain.

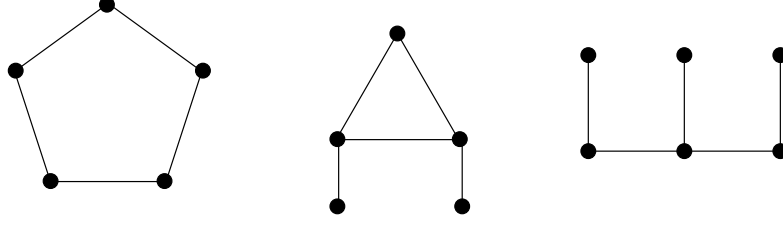


Figure 4: Stable graphs with 5 edges

**Claim 1**  $d(v_{i-2}) + d(v_i) + d(v_{i+2}) \geq 14$  holds for any  $i$ .

PROOF: Note that, the only way to make  $|N(v_i)| = 4$  is to add the edge  $\{v_i, v_{i-2}, v_{i+2}\}$  to  $\mathcal{H}$ , because  $N(v_i)$  already contains the edges  $\{v_{i-2}, v_{i-1}\}$ ,  $\{v_{i-1}, v_{i+1}\}$  and  $\{v_{i+1}, v_{i+2}\}$ .

Suppose that  $d(v_{i-2}) + d(v_i) + d(v_{i+2}) \leq 13$ . Since  $d(v_j) \geq 4$  for any  $j$ , there are only two cases.

If  $d(v_{i-2}) = d(v_i) = 4 \leq d(v_{i+2})$  (or  $d(v_{i-2}) \geq 4 = d(v_i) = d(v_{i+2})$ ) then  $\{v_{i-2}, v_i, v_{i+2}\} \in \mathcal{E}(\mathcal{H})$  must hold, but this implies  $d(v_{i-2}) \geq 5$ , a contradiction.

The other case is when  $d(v_{i-2}) = d(v_{i+2}) = 4 \leq d(v_i)$ . Since  $v_{i-2}$  and  $v_{i+2}$  is stable,  $\{v_{i-4}, v_{i-2}, v_i\}, \{v_i, v_{i+2}, v_{i+4}\} \in \mathcal{E}(\mathcal{H})$  holds. However, this means that  $N(v_i)$  contains a path of length 5 with 6 distinct vertices. This is a contradiction, because non of the stable graphs with 5 edges contains such a subgraph, therefore  $d(v_i) \geq 6$ .  $\square$

Using the above claim, we obtain that

$$9|\mathcal{E}(\mathcal{H})| = 3 \sum_{i=1}^n d(v_i) = \sum_{i=3}^{n+2} d(v_{i-2}) + d(v_i) + d(v_{i+2}) \geq 14n,$$

proving the theorem.  $\square$

**Theorem 7** *There exists a 2-edge-hamiltonian 3-uniform hypergraph  $\mathcal{H}$  on  $n$  vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{13}{4}n + o(n).$$

PROOF: The structure of the construction is very similar to that of Theorem 5. Let  $\mathcal{V}(\mathcal{H}) := \{w_1, \dots, w_p, v_1, \dots, v_q\}$  where  $p = \lceil n/4 \rceil$  and  $q = n - p$ . There are two types of edges in  $\mathcal{H}$ . The first kind of edges form a chain on  $\{v_1, \dots, v_q\}$ ,

$$\mathcal{E}_1(\mathcal{H}) := \{\{v_i, v_{i+1}, v_{i+2}\} \mid 1 \leq i \leq q\}.$$

The second kind connects the rest of the vertices to this chain:

$$\mathcal{E}_2(\mathcal{H}) := \left\{ \{w_i, v_{4(i-1)+j}, v_{4(i-1)+j+1}\} \mid 1 \leq i \leq p, 1 \leq j \leq 9 \right\}.$$

This means that the neighborhood of  $w_i$  is an ordinary graph, a path of length 9 formed by vertices  $v_{4(i-1)+1}, \dots, v_{5(i-1)+10}$ . The neighborhood of  $w_{i+1}$  is also a path of length 9, which begins at  $v_{4(i-1)+5}$ , so the neighborhood of  $w_i$  and  $w_{i+1}$  have 6 common vertices and the neighborhood of  $w_i$  and  $w_{i+2}$  have 2 common vertices (except maybe at the “end” where the overlap is larger if  $4 \nmid n$ ). Let  $\mathcal{E}(\mathcal{H}) := \mathcal{E}_1(\mathcal{H}) \cup \mathcal{E}_2(\mathcal{H})$ , then it is clear that  $|\mathcal{E}(\mathcal{H})| = q + 9p = n + 9\lceil n/4 \rceil = 13n/4 + o(n)$ . (See Fig. 5.)

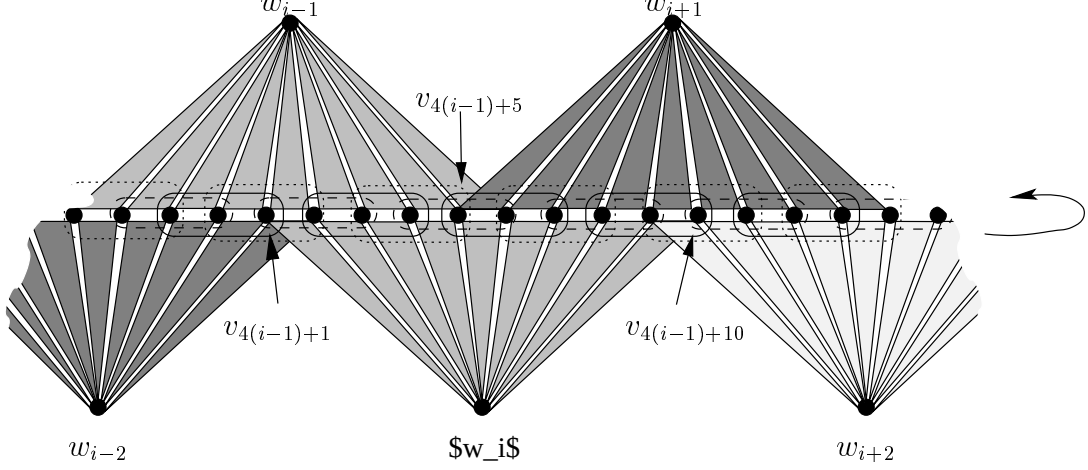


Figure 5: 2-edge-hamiltonian 3-uniform hypergraph

Using the method described in the proof of Theorem 5 it can be easily proven, that  $\mathcal{H}$  is 1-edge-hamiltonian. It is also clear that  $\mathcal{H}$  remains hamiltonian if the 2 removed edges are “far” from each other, namely if no  $w_i$  for which its neighborhood intersects both removed edges.

If both edges contains  $w_i$  then we can still insert  $w_i$  in a similar way as in Fig. 2, since there are 9 edges containing  $w_i$ , so after the removal of 2, we still have 3 consecutive.

The other cases can be also proved one by one, the reader may verify this with the help of a few examples on Fig. 6.

□

In order to obtain a lower bound for general  $k$ , one should know the minimum number of edges in a graph which contains a  $P_4$  after removing any  $k$  edges of the graph. These graphs are called  $k$ -stable and the minimum number of edges in a  $k$ -stable graph is denoted by  $S(k)$ .

A trivial upper bound is obtained for  $S(k)$  in the following way.

Observe that the maximum number of edges on  $n$  vertices in a  $P_4$ -free graph is  $n - 1$  if  $n$  is not divisible by 3 and  $n$  if  $n$  is divisible by 3. The extremal graphs are union of at least one star and some (possibly zero) triangles in the first case, and union of triangles in the second case. Taking the most dense graph on  $n$  vertices,

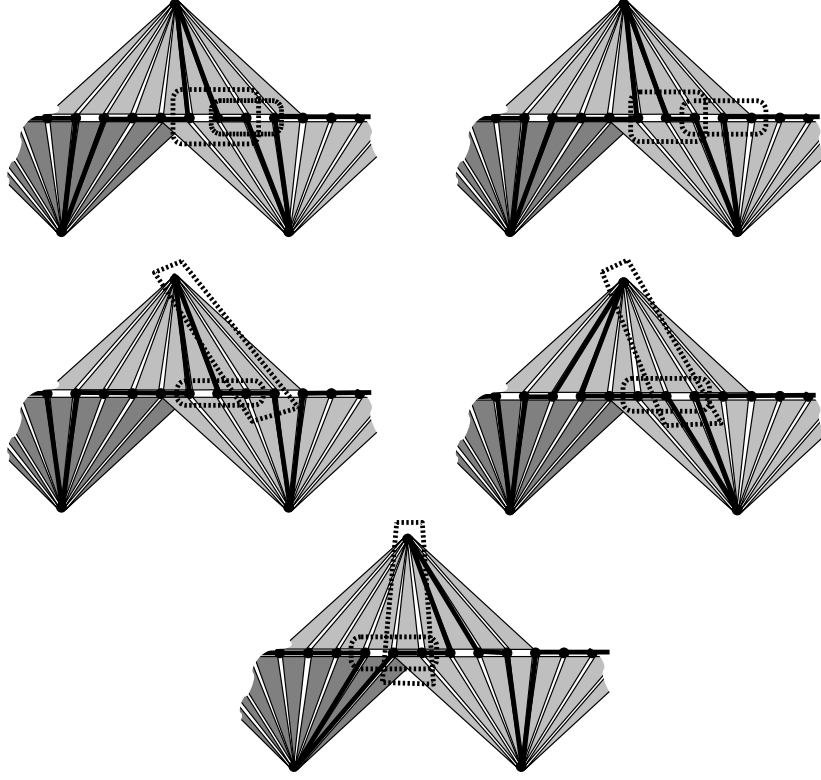


Figure 6: Examples of the more complicated cases

a complete graph or an almost complete graph will give the desired bound. By the above observation if  $e(G) > n - 1 + k$  if  $3 \nmid n$  and  $e(G) > n + k$  if  $3 \mid n$  then  $G$  is  $k$ -stable. However, for three values of  $k$  there are constructions which give bounds smaller by 1.

An other remark is that  $S(k)$  is strictly monotone, since by removing an edge from a  $k$ -stable graph results in a  $(k - 1)$ -stable graph.

The following lemma shows, that to prove  $S(k) > m$  it is enough to prove, that none of the graphs on exactly  $m$  edges are  $k$ -stable, so it is not possible that there is a  $k$ -stable graph with  $m - 2$  edges for example.

**Lemma 8** *If for any graph  $G$  with  $e(G) = m$  the graph is not  $k$ -stable then  $S(k) > m$ .*

PROOF: Let  $G'$  be a graph with  $e(G') < m$ . We will prove that  $G'$  cannot be  $k$ -stable. Construct  $G$  from  $G'$  by adding  $m - e(G')$  independent edges. Suppose indirectly that  $G'$  is  $k$ -stable. Since  $e(G) = m$ , if we remove  $k$  edges from  $G$  then we removed  $\leq k$  edges from  $G'$  so it will contain a  $P_4$ , thus  $G$  is  $k$ -stable, a contradiction.

□

The next lemma will help us to handle some easy extremal cases.

**Lemma 9** *If the maximum degree in  $G$  is 2, then  $G$  contains a  $P_4$ -free subgraph with at least  $\left\lceil \frac{e(G)}{2} \right\rceil$ .*

PROOF: It is enough to prove the claim for connected graphs, because otherwise taking the union of the subgraphs found in each component will prove the claim.

If the graph is an even cycle, then take every other edge for the desired subgraph. If the graph is an odd cycle, then first take two consecutive edges, then every other edges. Similarly is the graph is an odd path, then take every other edge starting with the first edge. If the graph is an even path, then take the first two consecutive edges and then every other edges.  $\square$

**Theorem 10**  $S(1) = 4, S(2) = 6$ .

PROOF: For  $k = 1$  the proof is trivial.

If  $k = 2$  then suppose that there exists a 2-stable graph  $G_2$  with 5 edges. It is clear that removing any edge of  $G_2$  gives a 1-stable graph with 4 edges, so it must be  $C_4$ . One can easily verify that there is no such  $G_2$ . Note that there are 3 different 2-stable graphs with 6 edges (see Fig. 7).

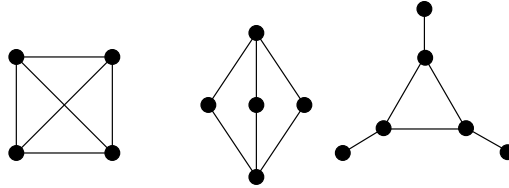


Figure 7: 2-stable graphs with 6 edges

On the other hand  $S(k) \leq 6$  since  $K_4$  is 2-stable. If we remove 2 edges from it 4 edges remain on 4 vertices, so it must contain a  $P_4$ .  $\square$

**Theorem 11**  $S(3) = 8, S(4) = 9, S(5) = 10$ .

PROOF: Since  $S(k)$  is strictly monotone, it is enough to prove that  $S(3) \geq 8$  and  $S(5) \leq 10$ .

To prove the second claim we show that  $K_5$  is 5-stable. If 5 edges are removed from  $K_5$  then 5 edges remain on 5 vertices, so it must contain a  $P_4$ .

To prove  $S(3) \geq 8$  suppose indirectly that there exists a 3-stable graph  $G_3$  with  $|E(G_3)| = 7$ .

- a) **There exists a vertex of degree  $\geq 4$  in  $G_3$ :** 4 edges incident to a vertex does not contain a  $P_4$  so if we remove the rest of the edges, no  $P_4$  remains. Thus we may suppose that the maximum degree in  $G_3$  is at most 3.



- b) **There exists a triangle in  $G_3$ :** If there is an edge independent from the triangle, then these 4 edges do not contain a  $P_4$ , so by removing the rest of the edges our claim is proved. Otherwise, all other edges have one end common with the triangle, moreover by case a) there is at most 1 such edge at each vertex of the triangle. This implies that  $|E(G_3)| \leq 6$ , so we may suppose that there is no triangle.
- c) **There exist a vertex  $v_1$  with degree 3:** Let  $v_2, v_3, v_4$  be its neighbors. Since there is no triangle in the graph, there are no edges between  $v_2, v_3, v_4$ . Since there are 4 more edges, there must be 2 of them which are not adjacent to 2 vertices of  $v_2, v_3, v_4$ , say to  $v_2, v_3$ . So these two edges and  $(v_1, v_2)$  and  $(v_1, v_3)$  forms a  $P_4$ -free subgraph. Thus we may suppose that the maximum degree is 2. Applying Lemma 9 we complete the proof.

□

**Theorem 12**  $S(6) = 12, S(7) = 13, S(8) = 14$ .

PROOF: Since  $S(k)$  is strictly monotone, it is enough to prove that  $S(6) \geq 12$  and  $S(8) \leq 14$ .

To prove the second claim we show that the graph  $G_8$  on Figure 8 is 8-stable. If 8 edges are removed from  $G_8$  then 6 edges remain on 7 vertices. There are only tree ways for these edges to form a  $P_4$ -free graph. 1) two independent triangles; 2) a triangle and a star with 3 edges; 3) a star with 6 edges.

1) is not possible, since any triangle contains at least two vertices from the  $a_i$  vertices. 2) is also not possible, since all stars with 3 edges in  $G_8$  contain at least two vertices from the  $a_i$  vertices, and the same holds for any triangle. 3) is not possible since there is no vertex with degree 6 in  $G_8$ .

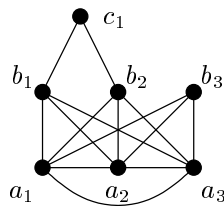


Figure 8:  $G_8$

To prove  $S(6) \geq 12$  suppose indirectly that there exists a 6-stable graph  $G_6$  with  $|E(G_6)| = 11$ .

- a) **There exists a vertex of degree  $\geq 5$  in  $G_6$ :** 5 edges incident to a vertex does not contain a  $P_4$  so if we remove the rest of the edges, no  $P_4$  remains. Thus we may suppose that the maximum degree in  $G_6$  is at most 4.

- b) **There exists a triangle in  $G_6$ :** If there are 2 independent edges from the triangle, then these 5 edges form a  $P_4$ -free subgraph. Otherwise, all other edges have one end common with the triangle, moreover by case a) there is at most 2 such edges at each vertex of the triangle. This implies that  $|E(G_6)| \leq 9$ , so we may suppose that there is no triangle.
- c) **There exist a vertex  $v_1$  with degree 4:** Let  $v_2, v_3, v_4, v_5$  be its neighbors. Since there is no triangle in the graph, there are no edges between  $v_2, v_3, v_4, v_5$ . Since there are 7 more edges, there must be 2 of them which are not adjacent to 3 vertices of  $v_2, v_3, v_4, v_5$ , say to  $v_2, v_3, v_4$ . So these two edges and  $(v_1, v_2), (v_1, v_3)$  and  $(v_1, v_4)$  forms a  $P_4$ -free subgraph. Thus we may suppose that the maximum degree is 3.
- d) **There exist a vertex  $v_1$  with degree 3:** Let  $v_2, v_3, v_4$  be its neighbors. Since there is no triangle in the graph, there are no edges between  $v_2, v_3, v_4$  and since the maximum degree is 3 there are at most 2 other edges incident to each of  $v_2, v_3, v_4$ . Thus there must be at least 2 edges which are not incident to any of  $v_1, v_2, v_3, v_4$ . So these two edges and  $(v_1, v_2), (v_1, v_3)$  and  $(v_1, v_4)$  forms a  $P_4$ -free subgraph. Thus we may suppose that the maximum degree is 2, so applying Lemma 9 we can complete the proof.

□

**Conjecture 13** *The minimum number of edges in a  $k$ -stable graph is*

$$S(k) = \left\lceil k + \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil + O(1).$$

Following theorem gives an upper bound on the maximum number of edges. We already know that this bound is better than the trivial one if  $2 \leq k \leq 8$  and if Conjecture 13 is true, than we obtain a good bound for larger  $k$  values, too.

**Theorem 14** *For any  $k$ -edge-hamiltonian 3-uniform hypergraph  $\mathcal{H}$  on  $n$  vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{S(k)}{3}n.$$

*holds.*

PROOF: If  $\mathcal{H}$  is  $k$ -hamiltonian then the neighborhood of any vertex must be  $k$ -stable, which implies that any vertex is contained by at least  $S(k)$  edges. Since every edge contains exactly 3 vertices, the claim is proved. □

### 3 1-edge-hamiltonian hypergraphs

**Theorem 15** *There exists a 1-edge-hamiltonian  $r$ -uniform hypergraph  $\mathcal{H}$  on  $n$  vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{4r-1}{2r}n + o(n).$$

PROOF: The idea of the construction is similar to the one on Fig. 1. Let  $\mathcal{V}(\mathcal{H}) := \{w_1, \dots, w_p, v_1, \dots, v_q\}$  where  $p = \lceil n/2r \rceil$  and  $q = n - p$ . There are two types of edges in  $\mathcal{H}$ . The first kind of edges form a chain on  $\{v_1, \dots, v_q\}$ ,

$$\mathcal{E}_1(\mathcal{H}) := \left\{ \{v_i, v_{i+1}, \dots, v_{i+r-1}\} \mid 1 \leq i \leq q \right\}.$$

The second kind connects the rest of the vertices to this chain:

$$\mathcal{E}_2(\mathcal{H}) := \left\{ \{w_i, v_{(2r-1)(i-1)+j}, \dots, v_{(2r-1)(i-1)+j+r-2}\} \mid 1 \leq i \leq p, 1 \leq j \leq 2r \right\}.$$

This means that the neighborhood of  $w_i$  is an  $(r-1)$ -uniform open chain of length  $2r$  formed by vertices  $v_{(2r-1)(i-1)+1}, \dots, v_{(2r-1)(i-1)+3r-2}$ . The neighborhood of  $w_{i+1}$  is also an open chain of length  $2r$ , which begins at  $v_{(2r-1)(i-1)+2r}$ , so

$$v_{5(i-1)+2r}, \dots, v_{5(i-1)+3r-2} \in N(w_i) \cap N(w_{i+1})$$

(except maybe for  $N(w_1)$  and  $N(w_p)$  where the overlap is larger if  $(2r) \nmid n$ ). Let  $\mathcal{E}(\mathcal{H}) := \mathcal{E}_1(\mathcal{H}) \cup \mathcal{E}_2(\mathcal{H})$ , then it is clear that  $|\mathcal{E}(\mathcal{H})| = q + 2rp = n + (2r-1)\lceil n/2r \rceil = \frac{4r-1}{2r}n + o(n)$ .

One can prove that this hypergraph is 1-hamiltonian in the same way as in Theorem 5.  $\square$

**Theorem 16** *For any 1-edge-hamiltonian 4-uniform hypergraph  $\mathcal{H}$  on  $n \geq 6$  vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{3}{2}n$$

*holds.*

PROOF: Following the idea of the proof of Theorem 6 we need to know what is the minimum number of edges in a 1-stable 3-uniform hypergraph. Now 1-stable means that the hypergraph contains an open chain with 4 edges on 6 vertices  $\mathcal{P}_6^{(3)}$ , since the edges of a hamiltonian-chain containing a fixed vertex form such an open chain.

It is easy to see that it is impossible to create a 1-stable hypergraph by adding only one edge to  $\mathcal{P}_6^{(3)}$ , therefore the minimum number of edges in a 1-stable hypergraph is 6, since the 3-uniform hyperchain on 6 vertices,  $\mathcal{C}_6^{(3)}$  is a 1-stable with 6 edges.

This gives that the minimum degree is 6, completing the proof.  $\square$

Note, that the above bound is already better than the trivial one. On the other hand, by case analysis, we can also prove that  $\mathcal{C}_6^{(3)}$  is the only 1-stable hypergraph with 6 vertices, which leads to a better lower bound:

$$|\mathcal{E}(\mathcal{H})| \geq \frac{11}{6}n$$

However, the proof is too long compared with the improvement, so it is omitted.

## 4 An application

**Theorem 17** *If an  $r$ -uniform hypergraph  $\mathcal{H}$  on  $n$  vertices has no hamiltonian chain then*

$$|\mathcal{E}(\mathcal{H})| \leq \binom{n}{r} \left( 1 - \frac{4r}{(4r-1)n} \right) \quad (1)$$

*holds.*

PROOF: Let  $m$  denote the number of missing edges (the  $r$ -element subsets which are not edges of  $\mathcal{H}$ ). By (1) we obtain

$$m < \frac{4r}{(4r-1)n} \binom{n}{r}.$$

Observe, that if a hypergraph contains a 1-edge-hamiltonian subgraph then one must delete at least 2 edges from it to destroy all hamiltonian chains. Therefore we count the number of occurrences of the 1-edge-hamiltonian hypergraph constructed in Theorem 15. Let  $\mathcal{G}$  denote this  $r$ -uniform hypergraph on  $n$  vertices.

It is a simple matter to prove that there are  $\frac{n!}{|\text{Aut}(\mathcal{G})|}$  different  $\mathcal{G}$  sub-hypergraphs in  $\mathcal{K}_n^{(k)}$ , where  $\text{Aut}(\mathcal{G})$  denotes the automorphism group of  $\mathcal{G}$ . Since every edges of  $\mathcal{K}_n^{(k)}$  is contained by the same number of  $\mathcal{G}$  sub-hypergraphs, the number of  $\mathcal{G}$  sub-hypergraphs which contains a specified edge is

$$\frac{|\mathcal{E}(\mathcal{G})|}{\binom{n}{r}} \cdot \frac{n!}{|\text{Aut}(\mathcal{G})|}.$$

Thus the number of hamiltonian chains in  $\mathcal{H}$  is

$$\geq 2 \cdot \frac{n!}{|\text{Aut}(\mathcal{G})|} - m \frac{|\mathcal{E}(\mathcal{G})|}{\binom{n}{r}} \cdot \frac{n!}{|\text{Aut}(\mathcal{G})|} > 0,$$

our claim is proved, because by Theorem 15  $|\mathcal{E}(\mathcal{G})| = \frac{4r-1}{2r}n$ .  $\square$

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