Searching for f-Hamiltonian circuits

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Let f be a continuous, monotone non-decreasing, non-negative function. Let G be a graph with n vertices. A circuit K of G is called f–**Hamiltonian** if the distance of every vertex of G from some vertex of K is at most $f(n)$. An f–**Hamiltonian path** is defined analogously.

Deciding whether a graph has an f–Hamiltonian circuit, a problem posed by G. O. H. Katona, is NP–complete for a general f (put $f \equiv 0$). We prove NP–completeness for the case $f(n) \leq n^{1-\varepsilon}$ (Theorem 1) and polynomial solvability for $f(n) \geq n/c$ (Theorem 2). These results hold for f–Hamiltonian paths as well. However, f–Hamiltonian paths always exist if $f(n) \geq \frac{n-4}{3}$, on Theorem 3.

THEOREM 1. *The problem is NP-complete if* $0 < f(n) \leq n^{1-\varepsilon}$ *holds for some* ε *and for all* $n \geq n_0$.

LEMMA 1. *If the above function* f *and the above numbers* ε, n_0 *are given then, for any positive integer* h *there exist some non-negative integers* s *and* u *satisfying* $f(n) \leq n/\sqrt[k]{n} = n^{1-\varepsilon}$ *for all* $n > n_0$ *there exist non-negative integers* s *and* u *such that* $\lfloor f\big((s+1)h+u\big)\rfloor = s, \quad s \leq 2^k h^{k-1}$ *and* $u \leq 2^k h^{k-2}$.

PROOF (of Lemma 1): Consider the function $g(t) = f(t) - (\frac{t}{h} - 1)$. Clearly, $g(0) > 0$ and $g(t) < 0$ if t is large enough since $f(t) \leq t/\sqrt[k]{t} \leq \frac{t}{h} - 1$. Let t_0 be the smallest number satisfying $g(t_0) = 0$; such numbers exist by Bolzano's theorem. Let us investigate the behavior of the functions $f(t)$ and $\frac{t}{h} - 1$ in the interval $[\lfloor t_0 \rfloor, t_0]$. Observe first that $\lfloor \frac{t_0}{h} - 1 \rfloor = \lfloor \frac{\lfloor t_0 \rfloor}{h} - 1 \rfloor$, for otherwise $\frac{t}{h} - 1$ should by its monotonicity be integer at an internal point of the interval.

Applying the monotonicity of $f(t)$ and the relation $\frac{t}{h} - 1 \leq f(t)$ in this interval we obtain $\lfloor \frac{\lfloor t_0 \rfloor}{k} - 1 \rfloor \leq \lfloor f(\lfloor t_0 \rfloor) \rfloor \leq \lfloor f(t_0) \rfloor = \lfloor \frac{t_0}{h} - 1 \rfloor$.

Now define s and u by $\quad \lfloor t_0 \rfloor = (s+1)h + u \quad (0 \leq u < h)$. Then $\lfloor f\big((s+1)h+u\big)\rfloor = s$ clearly follows. In order to see the remaining properties, suppose indirectly that $t_0 > 2^k h^k, n_0$. Then

$$\frac{t_0}{\sqrt[k]{t_0}} \geq f(t_0) = \frac{t_0}{h} - 1 > \frac{t_0}{\frac{1}{2}\sqrt[k]{t_0}} - 1$$

implies

$$\frac{t_0}{\sqrt[k]{t_0}} \leq 1$$

a contradiction. ■

PROOF (of Theorem 1): We reduce the problem „Decide whether a given graph H is Hamiltonian" to our present one. Let H contain h vertices; choose s and u as in Lemma 1. Attach a path of length s each to every vertex of H and additional u paths of length one to one distinguished vertex (see Figure 1).

The new graph G has $(s+1)h+u$ vertices (which is a polynomial function of h). We claim that G is f–Hamiltonian if and only if H was Hamiltonian. The if part is obvious since $\lfloor f\big((s+1)h+u\big)\rfloor = s$. For the only if part all we have to observe is that an f–Hamiltonian circuit K of G is a circuit, hence does not contain any of the new edges and if K did not contain a vertex x of H then the other end of the path attached to x would violate f–Hamiltonicity. ■

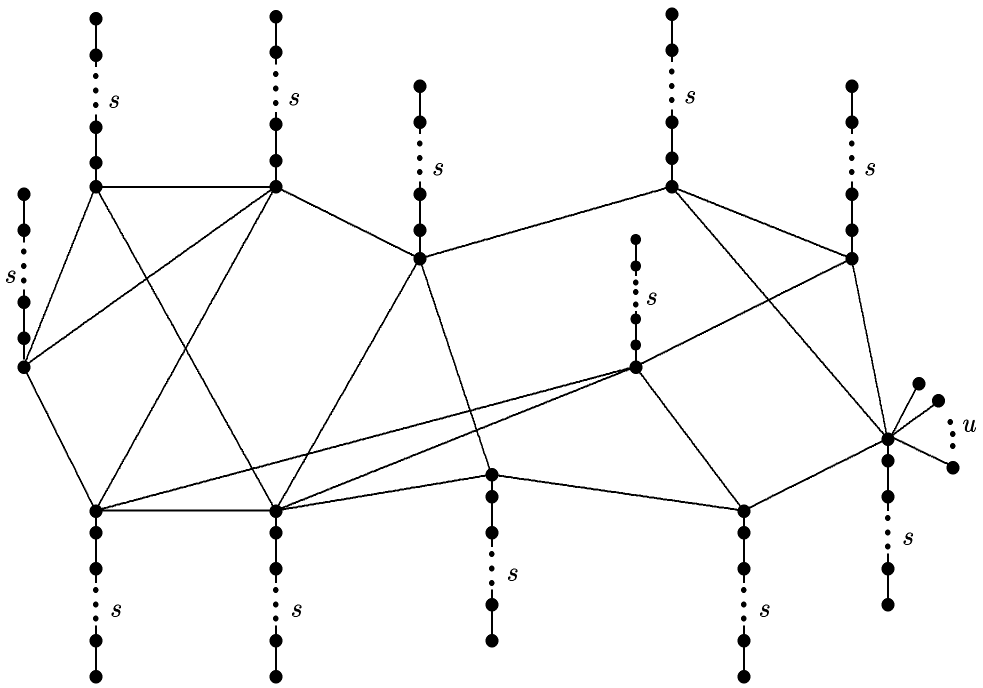

Figure 1.

THEOREM 2. *The problem is polynomially solvable if $f(n) \geq n/c$ holds for some positive integer c and for all $n > n_0$.*

LEMMA 2. *The graph G has an f–Hamiltonian circuit K if and only if there is a set R of at most $c+1$ vertices, contained in a circuit C so that the distance of every vertex of G from some vertex of R is at most $f(n)$.*

PROOF (of Lemma 2): The „if" part is obvious since C can play the role of K. Conversely, we construct R from K. Let $k_1, k_2, \ldots, k_t$ be the vertices of K in this order. For every vertex v of G , determine the closest vertex k_i of K. Denote by K_i the set of those vertices of G which are closest to k_i. The relation $k_i \in K_i$ is obvious. After this we build the breadth first spanning tree of the subgraph induced by K_i from k_i, as the root of the tree. First we connect k_i with the vertices which are adjacent to k_i in G. Then we connect one of these vertex with the vertices adjacent to it in G, but not connected earlier to any other vertex. And so on, until we have connected every vertex to the tree. Since the graph is connected we connect every point to some other one. Obviously we obtain a tree F_i for each i. Let F denote the union of the circuit K and the trees F_i (see Figure 2).

These trees determine the shortest paths between the circuit and the vertices of the graph. Thus it is easy to determine which vertex of F_i is the farthermost from the root k_i. Denote this vertex by q_i and let μ_i its distance from k_i. Define

$$\omega_1 = \max_{1 \leq j \leq t} \{j| \quad \mu_i + j - i \leq f(n) \quad \text{for every} \quad 1 \leq i \leq j\}.$$

This means that k_{ω_1} has the property, that if $i \leq \omega_1$ then none of the vertices of F_i is farther from k_{ω_1} than $f(n)$ in the graph G, since there is such a path between k_{ω_1} and the vertices of F_i. On the other hand there exists a vertex among the vertices of F_i for some $i \leq \omega_1$ such that its distance from k_{ω_1} is exactly $f(n)$ in F. Now cancel the vertices of the trees F_i from

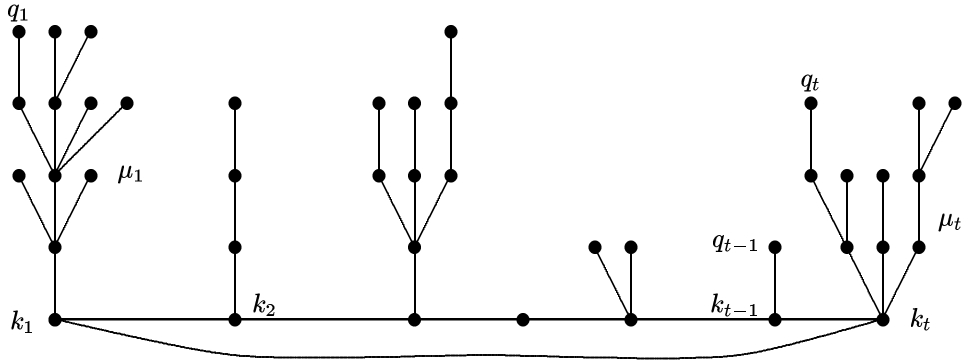

Figure 2.

the graph F. For the remaining graph, determine ω_2 exactly like ω_1 before. Repeating this step, we obtain the subscripts $\omega_3, \ldots, \omega_m$. All the statements for k_{ω_1} are true for every k_{ω_i}, too, except for k_{ω_m}, for which it is not sure that there exists a vertex with distance exactly $f(n)$ from it. Thus $R = \{k_{\omega_1}, k_{\omega_2}, \ldots, k_{\omega_m}\}$ satisfies the conditions that every vertex of G can be reached from one of the vertices of the set R on a path not longer than $f(n)$, and there is a circuit in G , namely K, which contains all the vertices of the set R.

Thus there exist vertex disjoint paths of length exactly $f(n)$ starting from $k_{\omega_1}, k_{\omega_2}, \ldots, k_{\omega_{m-1}}$, resp. So the number of the vertices in the graph G is at least $(m-1)f(n)$. Hence we have $(m-1)f(n) \leq n$. The condition $f(n) \geq n/c$ implies $m \leq c+1$. ■

PROOF (of Theorem 2.): Let us check whether the condition of Lemma 2 holds. Recall that our upper bound for the number of steps must be a polynomial in the number n of the vertices but the coefficients of this polynomial may contain c in the exponent. If we choose a set $d \leq c+1$ vertices arbitrarily we can easily check whether every vertex can be reached from these ones along a path of length at most $f(n)$. All we have to check is whether there is a circuit containing all the vertices of R.

Robertson and Seymour [1] proved that if $p_1, p_2, \ldots, p_d, q_1, q_2, \ldots, q_d$ are given vertices of a graph then one can decide in polynomial time whether there are d vertex disjoint paths connecting p_1 to q_1, p_2 to q_2 and so on. Apply this algorithm with $p_1 = v_1, q_1 = p_2 = v_2, q_2 = p_3 = v_3, \ldots, q_d = v_1$ where $v_1, v_2, \ldots, v_d$ are the elements of R (in some order). Since we can choose R in any way and try its elements in any order, our algorithm is polynomial. ■

The theorems are true for f–Hamiltonian paths as well, with slight modifications in the proofs. The following additional result may be of some interest.

THEOREM 3. *If $f(n) > \frac{n-4}{3}$, then there is an f–Hamiltonian path in the graph.*

PROOF: Let $t_1, t_2, \ldots, t_d$ be the longest simple path in the graph. We prove that this is an f–Hamiltonian path. Assume indirectly that there is a vertex p which is not reachable from this path along a path not longer than $f(n)$. Let t_i be the closest vertex to p from the vertices of the path. This means that the length l of the shortest path from t_i to p satisfies $l > f(n)$. Suppose that $i-1 < l$. Then the simple path from p to t_d via t_i is longer than $(t_1, \ldots, t_d)$, a contradiction. We obtained $i-1 \geq l$. The inequality $d-i \geq l$ is obtained in the same way. The total number of vertices along these paths is $d+l \geq (i-1)+(d-i)+1+l \geq 3l+1 \geq 3(f(n)+1)+1$. This contradicts our assumption $\frac{n-4}{3} < f(n)$. ■

The two examples of Figure 3 with $n = 10, f(10) = 2$ show that the algorithm is really

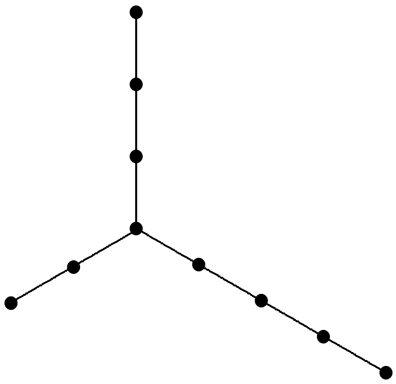

Figure 3/a

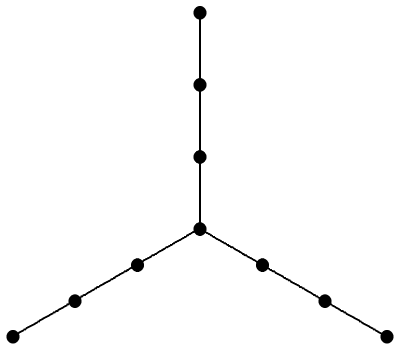

Figure 3/b

needed if $f(n) \leq \frac{n-4}{3}$.

The remaining question is the case

$$\lim_{n\to\infty} f(n)/n = 0 \quad \text{and} \quad \lim_{n\to\infty} \sqrt[k]{n} f(n)/n > 1$$

for every positive k.

I would like to thank András Frank for calling my attention to the paper of Robertson and Seymour, and András Recski for suggestions in styling of my paper.

Reference

[1] *Robertson*, N.; *Seymour,* P.D.: *An outline of a Disjoint path algorithm*; in preparation