Properties of edge–tough graphs

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Abstract. In [6] the author defined a new property of graphs namely the edge–toughness. It was proved in [6] that a $2t$–tough graph is always t–edge–tough. It is proved in the present paper that this is not true for $(2t-\varepsilon)$–tough graphs if t is a positive integer. A result of Enomoto et al. in [5] implies that every 2–tough graph has a 2–factor. In the present paper it is proved that every 1–edge–tough graph has a 2–factor. This is a sharpening of the previous statement.

1. Introduction

In the present paper only finite, undirected and simple graphs are considered. In 1973 Chvátal [3] introduced the definition of toughness. Many authors investigated the relation of toughness and hamiltonicity since Chvátal's paper. A good survey of the topic is [2]. Let $\omega(G)$ denote the number of components of the graph G. If $T \subseteq V(G)$ then the graph $G-T$ has vertices $V(G-T) = V(G) - T$ and $(u,v) = e \in E(G-T)$ if and only if $e \in E(G)$ but neither u nor v is in T. A graph G is *t–tough* if $|S| \geq t\omega(G-S)$ for every $S \subseteq V(G)$ with $\omega(G-S) > 1$. The *toughness* of G, denoted by $t(G)$, is the maximum value of t for which G is t-tough (taking $t(K_n) = \infty$ for all $n \geq 1$). A *k–factor* of a graph is a k–regular spanning subgraph. Finally if $v \in V(G)$ then $N(v) = \{u \in V(G) | (v,u) \in E(G)\}$ denotes the neighborhood of v.

Chvátal investigated the relationship between toughness and hamiltonicity in [3]. Being 1–tough is clearly a necessary condition for a graph to be hamiltonian. He conjectured that there exists a finite constant t_0 such that every t_0–tough graph is hamiltonian. This conjecture is still open. Enomoto, Jackson, Katerinis and Saito in [5] showed that if there is such a t_0 then $t_0 \geq 2$. In fact this is a consequence of the following theorem.

Theorem 1. [Enomoto et al.] *Let $k \geq 1$. For any $\varepsilon > 0$ there exists a $(k-\varepsilon)$–tough graph G on n vertices with $n \geq k+1$ and kn even which has no k–factor.*

For $k = 2$ this means that $(2-\varepsilon)$–tough graphs were shown which have no 2-factor and therefore they are not hamiltonian.

On the other hand they also proved in [5] that this result is sharp.

Theorem 2. [Enomoto et al.] *Let G be a k–tough graph on n vertices with $n \geq k+1$ and kn even. Then G has a k–factor.*

In [6] the author defined a new property of graphs. When the toughness is defined only vertices are to be deleted. Now edges are also allowed to be deleted besides the vertices.

For a graph G, let $X \subseteq V(G)$ and $Y \subseteq E(G-X)$. Let $Q_1, Q_2, \ldots, Q_n$ be the components of the edge–induced subgraph $G(Y)$ induced by Y. The vertices of $G(Y)$ are those vertices of G which are incident to an edge in Y and the edges of $G(Y)$ are those edges of G whose endvertices are both in the vertex set of $G(Y)$. Therefore there may be edges in $G(Y)$ which are not in Y. Let H be a graph and B a subgraph of H. The boundary of B in H is defined as $\mathrm{bd}_H(B) = \{v \in V(B) | v \text{ has a neighbor outside of } B\}$ and $\mathrm{in}_H(B) = V(B) - \mathrm{bd}_H(B)$. Therefore, in our case

$$\mathrm{bd}_{G-X}(Q_i) = \{v \in V(Q_i) \mid v \text{ has a neighbor outside of } Q_i \text{ and } X\}$$

and

$$\mathrm{in}_{G-X}(Q_i) = V(Q_i) - \mathrm{bd}_{G-X}(Q_i).$$

$G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)$ denotes the graph which is obtained by deleting the vertices of X and $\mathrm{in}_{G-X}(Q_i)$ and the edges of Y from G (but not deleting the endvertices of these edges).

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Definition 1. *Let t be a positive number. G is t–**edge–tough** if G is connected and*

$$\omega\left(G - X - Y - \bigcup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) \le \frac{|X| + \sum_{i=1}^{n} \left\lfloor \frac{|\mathrm{bd}_{G-X}(Q_i)|}{2} \right\rfloor}{t} =: \frac{s(X,Y;G)}{t}$$

holds for every $X \subseteq V(G)$ and $Y \subseteq E(G-X)$ satisfying $\omega\left(G - X - Y - \cup_{i=1}^{n}\mathrm{in}_{G-X}(Q_i)\right) > 1$, using the notation introduced above. The **edge–toughness** *of G, denoted by $t_e(G)$, is the maximum value of t_e for which G is t_e–edge–tough (taking $t_e(K_n) = \infty$ for all $n \ge 1$).*

The following theorems were proved in [6] by the author.

Theorem 3. *If G is t–edge–tough then it is t–tough.*

Theorem 4. *If G is hamiltonian then it is 1–edge-tough.*

Theorem 5. *If G is $2t$–tough then it is t–edge–tough.*

Theorem 6. *For every $1 \ge \varepsilon > 0$ and every integer $\delta \ge 4$ there exists a $(2-\varepsilon)$–tough but non–1–edge–tough graph with minimum degree δ.*

In Section 2 Theorem 6 is generalized. In Section 3 we sharpen the special case $k = 2$ of Theorem 2. The following theorem is a corollary of a more general result of Edmonds [4]. It will be used in the proof of the latter result.

Theorem 7. [Edmonds] *G has a 2–factor if and only if*

$$|V'| \ge \frac{1}{2}\left(|K| + \mu\right)$$

holds for all $V' \subseteq V(G)$, where K denotes the set of vertices having degree at most one in $G - V'$ and μ denotes the number of components of $G - V'$ containing an odd number of vertices in K.

2. $(2t-\varepsilon)$–tough and non–t–edge–tough graphs

Theorem A. *For any positive integer t and any positive real number $\varepsilon < 1$ there exist $(2t-\varepsilon)$–tough but non–t–edge–tough graphs.*

Proof: By construction. Let p be a positive integer such that $p \ge \frac{2t}{\varepsilon}$. The graph $G_{t,p}$ satisfying the conditions of the theorem is given below.

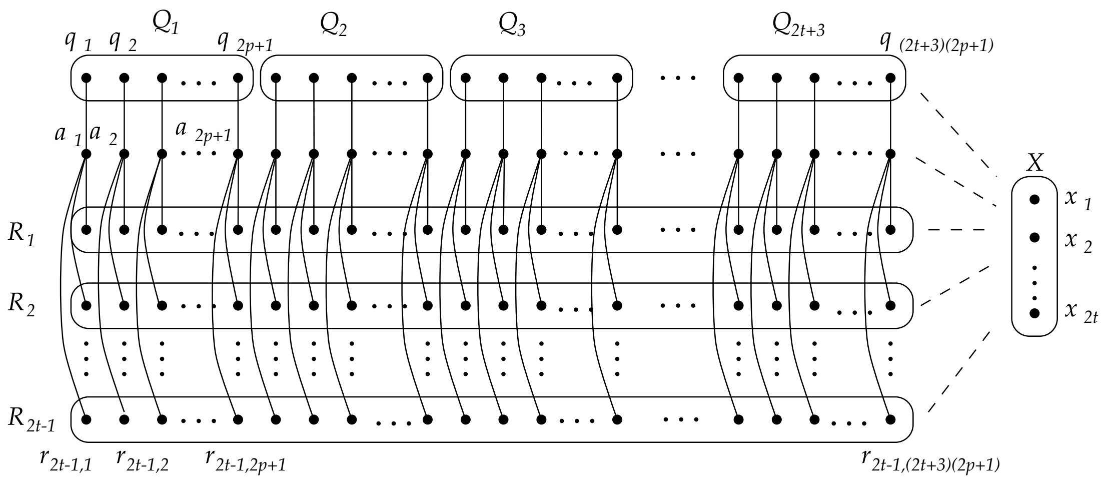

Figure 1

The vertex set $V(G_{t,p})$ of the graph is the union of the following sets, see Figure 1.

$$\begin{aligned}
X &= \{x_1, x_2, \ldots, x_{2t}\},\\
A &= \{a_1, a_2, \ldots, a_{(2t+3)(2p+1)}\},\\
Q_i &= \{q_{(2p+1)(i-1)+1}, q_{(2p+1)(i-1)+2}, \ldots, q_{(2p+1)(i-1)+2p+1}\} \text{ for } 1 \leq i \leq (2t+3),\\
R_i &= \{r_{i,1}, r_{i,2}, \ldots, r_{i,(2t+3)(2p+1)}\} \text{ for } 1 \leq i \leq (2t-1).
\end{aligned}$$

In order to have shorter formulas we denote $\alpha = (2t+3)(2p+1) = 4tp + 6p + 2t + 3$, $\mathcal{Q} = \cup_1^{2t+3} Q_i$ and $\mathcal{R} = \cup_1^{2t-1} R_i$. Thus $V(G_{t,p}) = X \cup A \cup \mathcal{Q} \cup \mathcal{R}$.

The edges of $G_{t,p}$ are given next. All vertices of X are adjacent to all other vertices of $G_{t,p}$ including all other vertices of X. For each $1 \leq i \leq (2t+3)$, Q_i spans a complete graph K_{2p+1}. For each $1 \leq i \leq (2t-1)$, R_i spans a complete graph K_α. For each $1 \leq i \leq (2t+3)(2p+1)$, the neighborhood of a_i is defined by $N(a_i) = \{q_i, r_{1,i}, r_{2,i}, \ldots, r_{2t-1,i}\} \cup X$. Note that A spans an empty graph and that there are no edges between R_i and R_j if $i \neq j$ and between $\mathcal{Q}$ and $\mathcal{R}$.

First we prove that $G_{t,p}$ is not t–edge–tough. Since it will not cause a misunderstanding, we henceforth use G instead of $G_{t,p}$. Let Y be the set of edges spanned by $\mathcal{Q}$ and $\mathcal{R}$. Thus $G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)$ consists of disjoint stars spanned by $\{a_i, q_i, r_{1,i}, \ldots, r_{2t+1,i}\}$, where $1 \leq i \leq \alpha$.

Therefore $\omega\left(G - X - Y - \cup_{i=1}^{n} \mathrm{in}_{G-X}(Q_i)\right) = \alpha$. The components of the edge-induced subgraph $G(Y)$ are $Q_1, Q_2, \ldots, Q_{2t+3}, R_1, R_2, \ldots, R_{2t-1}$. Also $\mathrm{bd}_{G-X}(Q_i) = Q_i$ and $\mathrm{bd}_{G-X}(R_i) = R_i$. Thus $|\mathrm{bd}_{G-X}(Q_i)| = 2p+1$, $|\mathrm{bd}_{G-X}(R_i)| = \alpha$ and $\mathrm{in}_{G-X}(Q_i) = \mathrm{in}_{G-X}(R_i) = \emptyset$.

It must be proven that

$$\omega\left(G - X - Y\right) = \alpha > \frac{2t + (2t+3)\left\lfloor \frac{2p+1}{2} \right\rfloor + (2t-1)\left\lfloor \frac{\alpha}{2} \right\rfloor}{t} = \frac{s(X,Y;G)}{t}.$$

This is equivalent to

$$t(2t+3)(2p+1) > 2t + (2t+3)p + (2t-1)(2pt + 3p + t + 1)$$

which is clearly true.

To complete the proof we have to prove that G is $(2t-\varepsilon)$–tough. We prove this by contraposition. Suppose that there exists a subset of vertices S such that $(2t-\varepsilon)\omega(G-S) > |S|$ and $\omega(G-S) > 1$. The latter condition implies $X \subseteq S$.

Let $\mathcal{C}_{\mathrm{a}}$ be the set of those components of $G-S$ which contain vertices only from A. This implies that these components are isolated vertices since there are no two connected vertices in A. Let $\mathcal{C}_{\mathrm{b}}$ denote the set of the rest of the components of $G-S$. Each of these components must contain a vertex from $V(G) - A$. Finally denote $|\mathcal{C}_{\mathrm{a}}|$ and $|\mathcal{C}_{\mathrm{b}}|$ by c_{a} and c_{b}, respectively. Thus $c_{\mathrm{a}} + c_{\mathrm{b}} = \omega(G-S)$.

Case 1. Suppose that $0 \leq c_{\mathrm{b}} \leq 1$. If a_i is an isolated vertex in $G-S$ then $N(a_i) \subseteq S$. Since $N(a_i) \cap N(a_j) = \emptyset$ if $i \neq j$ we obtain $|S| \geq |X| + 2tc_{\mathrm{a}}$, because $|N(a_i)| = 2t$. Therefore

$$|S| \geq 2t + 2tc_{\mathrm{a}} \geq 2t(1 + c_{\mathrm{a}}) \geq 2t\omega(G-S) > (2t-\varepsilon)\omega(G-S),$$

a contradiction.

Case 2. This case covers all the other cases, in which $c_{\mathrm{b}} \geq 2$ and $c_{\mathrm{a}} \geq 0$. There are 3 subcases:

Case 2.1. There exist 2 different components of $G-S$ which contain at least one vertex of $\mathcal{R}$.

Case 2.2. Exactly one component of $G-S$ contains vertices of $\mathcal{R}$.

Case 2.3. No component of $G-S$ contains any vertex of $\mathcal{R}$.

Case 2.1. Two vertices of the same Q_i or R_i cannot be in different components of $G-S$ thus

$$\omega(G-S) \le c_{\mathrm{a}} + (2t+3) + (2t-1) = c_{\mathrm{a}} + 4t + 2.$$

As in Case 1, S must contain $N(a_i)$ if a_i is a component in $G-S$. Let S_1 be the union of these neighborhoods. This implies $S_1 \subseteq S$. It is easy to see that $|S_1| = 2tc_{\mathrm{a}}$.

Let $r_{n,m}, r_{k,l} \in V(G-S)$ where $n \neq k$ and such that they are in different components of $G-S$. The paths $(r_{n,m}, r_{n,i}, a_i, r_{k,i}, r_{k,l})$ for $1 \le i \le \alpha$ are vertex disjoint paths each connecting $r_{n,m}$ and $r_{k,l}$. If two consecutive vertices of any of these paths coincide they are replaced by one of them. Since these two vertices are in different components of $G-S$, S must contain at least one vertex from each of these paths. Let S_2 denote a set of vertices that contains exactly one vertex from each path. Thus $|S_2| = \alpha$. To have a lower bound on $|S|$ we need an upper bound on $|S_1 \cap S_2|$. By the construction of the graph, $N(a_i)$ and $(r_{n,m}, r_{n,i}, a_i, r_{k,i}, r_{k,l})$ have two common vertices but S_2 may contain at most one of them. This implies that each different neighborhood may have at most one vertex in S_2 which gives the conclusion $|S_1 \cap S_2| \le c_{\mathrm{a}} = \frac{|S_1|}{2t}$.

Since $X \subset S$, we obtain

$$|S| \ge |X| + |S_1| + |S_2| - |S_1 \cap S_2| \ge |X| + \left(1 - \frac{1}{2t}\right)|S_1| + |S_2| = 2t + (2t-1)\,c_{\mathrm{a}} + (2t+3)(2p+1).$$

Therefore, by assumption

$$(2t-\varepsilon)\,(c_{\mathrm{a}} + 4t + 2) > 2t + (2t-1)\,c_{\mathrm{a}} + (2t+3)(2p+1).$$

which is equivalent to

$$(1-\varepsilon)\,c_{\mathrm{a}} + (2t-\varepsilon)\,(4t+2) - 2t > (2t+3)(2p+1).$$

Since $c_{\mathrm{a}} \le \alpha = (2t+3)(2p+1)$ and $1-\varepsilon > 0$ we obtain

$$(1-\varepsilon)\,(2t+3)(2p+1) + (2t-\varepsilon)\,(4t+2) - 2t > (2t+3)(2p+1).$$

This implies

$$\frac{(2t-\varepsilon)\,(4t+2) - 2t}{2\varepsilon(2t+3)} - \frac{1}{2} > p.$$

It is easy to see that the left hand side is less than $\frac{2t}{\varepsilon}$, contradicting our assumption on p.

Case 2.2. Without loss of generality we may suppose that $G-S$ contains $r_{1,1}$. As a consequence all the other $c_{\mathrm{b}} - 1$ components of $\mathcal{C}_{\mathrm{b}}$ must contain a vertex from $\mathcal{Q}$. Let $q_{(2p+1)k+l}$ be such a vertex. The paths $(r_{1,1}, r_{1,(2p+1)k+i}, a_{(2p+1)k+i}, q_{(2p+1)k+i}, q_{(2p+1)k+l})$ for $1 \le i \le 2p+1$ are vertex disjoint paths, apart from the endvertices, each connecting $r_{1,1}$ and $q_{(2p+1)k+l}$. Again, if two consecutive vertices of any of these paths coincide they are replaced by one of them. Therefore S must contain at least one vertex from each of these $2p+1$ paths. This is true for all other components which contain a vertex from $\mathcal{Q}$ and the paths obtained this way are vertex disjoint from each other, apart from the endvertices. Therefore there are $(c_{\mathrm{b}}-1)(2p+1)$ vertex disjoint paths, each connecting $r_{1,1}$ and a vertex of $\mathcal{Q}$. Thus S must contain at least one vertex from each of these paths.

Let S_3 denote a subset of S which contains exactly one vertex from each of these paths. It is easy to see that such a set S_3 exists and $|S_3| = (c_{\mathrm{b}}-1)(2p+1)$.

As in Case 1, S must contain $N(a_i)$ if a_i is a component in $G-S$. Let S_4 be the union of these neighborhoods. This implies $S_4 \subseteq S$. It is easy to see that $|S_4| = 2tc_{\mathrm{a}}$.

To continue the proof we need a lower bound on $|S_3 \cup S_4|$. By the construction of the graph, $N(a_{(2p+1)k+i})$ and $(r_{1,1}, r_{1,(2p+1)k+i}, a_{(2p+1)k+i}, q_{(2p+1)k+i}, q_{(2p+1)k+l})$ have two common vertices but S_3 may contain at most one of them. This implies that each different neighborhood may have at most one vertex in S_3, which gives the conclusion $|S_3 \cap S_4| \le \frac{|S_4|}{2t}$. Thus $|S_3 \cup S_4| \ge |S_3| + \left(1 - \frac{1}{2t}\right)|S_4|$ holds.

On the other hand $q_{(2p+1)k+l} \in V(G-S)$, and thus $N(a_{(2p+1)k+l}) \not\subseteq S$. This implies that $r_{1,(2p+1)k+l} \in S_3 - S_4$ or $a_{1,(2p+1)k+l} \in S_3 - S_4$ because $(r_{1,1}, r_{1,(2p+1)k+l}, a_{(2p+1)k+l}, q_{(2p+1)k+l})$ is a path connecting $r_{1,1}$ and $q_{(2p+1)k+l}$. Since the same argument works for all the other components of $\mathcal{C}_{\mathrm{b}}$, except the one which contains $r_{1,1}$, this implies $|S_3 - S_4| \geq c_{\mathrm{b}} - 1$. Therefore $|S_3 \cup S_4| \geq |S_4| + c_{\mathrm{b}} - 1$ also holds. Thus

$$|S_3 \cup S_4| \geq \max\left(|S_3| + \left(1 - \frac{1}{2t}\right)|S_4|, |S_4| + c_{\mathrm{b}} - 1\right) =$$
$$= |S_4| + \max\left(|S_3| - \frac{1}{2t}|S_4|, c_{\mathrm{b}} - 1\right).$$

Two cases are now distinguished:
Case 2.2.1. $|S_3| - \frac{1}{2t}|S_4| \geq c_{\mathrm{b}} - 1$
Case 2.2.2. $|S_3| - \frac{1}{2t}|S_4| < c_{\mathrm{b}} - 1$.

Case 2.2.1. In this case by assumption we obtain

$$(2t-\varepsilon)\,\omega(G-S) = (2t-\varepsilon)\,(c_{\mathrm{a}} + c_{\mathrm{b}}) > |X| + |S_3 \cup S_4| \geq 2t + 2tc_{\mathrm{a}} + (c_{\mathrm{b}} - 1)(2p+1) - c_{\mathrm{a}}. \tag{1}$$

Let $m = |S_3| - \frac{1}{2t}|S_4| = (c_{\mathrm{b}} - 1)(2p+1) - c_{\mathrm{a}}$. In this case $m \geq c_{\mathrm{b}} - 1$, and thus $m > 0$ holds. Hence c_{b} can be expressed as a function of m:

$$c_{\mathrm{b}} = \frac{m + c_{\mathrm{a}}}{2p+1} + 1.$$

Substituting this into (1) we obtain

$$(2t-\varepsilon)\left(c_{\mathrm{a}} + \frac{m + c_{\mathrm{a}}}{2p+1} + 1\right) > 2t + 2tc_{\mathrm{a}} + m$$

or, equivalently,

$$(2t-\varepsilon)\,c_{\mathrm{a}}\,(2p+2) + (2t-\varepsilon)\,m + (2t-\varepsilon)\,(2p+1) > 2t(2p+1) + 2tc_{\mathrm{a}}(2p+1) + m(2p+1). \tag{2}$$

Our assumption on p ensures that $(2t-\varepsilon)\,m < m(2p+1)$. It is easy to see that $(2t-\varepsilon)\,(2p+1) < 2t(2p+1)$. With these two observations we obtain

$$(2t-\varepsilon)\,c_{\mathrm{a}}\,(2p+2) > 2tc_{\mathrm{a}}(2p+1)$$

from (2). This is a contradiction if $c_{\mathrm{a}} = 0$. If $c_{\mathrm{a}} \geq 1$ then this implies $\frac{t}{\varepsilon} > \frac{t-\varepsilon}{\varepsilon} > p$, contradicting our assumption on p.

Case 2.2.2. In this case by assumption we obtain

$$(2t-\varepsilon)\,\omega(G-S) = (2t-\varepsilon)\,(c_{\mathrm{a}} + c_{\mathrm{b}}) > |X| + |S_3 \cup S_4| \geq 2t + 2tc_{\mathrm{a}} + c_{\mathrm{b}} - 1$$

or, equivalently,

$$c_{\mathrm{b}}(2t - 1 - \varepsilon) - 2t + 1 > c_{\mathrm{a}}\varepsilon. \tag{3}$$

The assumption of this case implies $(c_{\mathrm{b}} - 1)2p < c_{\mathrm{a}}$, so from (3) we obtain

$$c_{\mathrm{b}}(2t - 1 - \varepsilon) - 2t + 1 > (c_{\mathrm{b}} - 1)2p\varepsilon.$$

This implies

$$t > \frac{1}{2}(2t - 1 - \varepsilon) - \frac{\varepsilon}{2c_{\mathrm{b}} - 2} > p\varepsilon,$$

contradicting our assumption on p.

Case 2.3. In this case each component of $\mathcal{C}_{\mathrm{b}}$ must contain a vertex from $\mathcal{Q}$ since they cannot contain vertices of $\mathcal{R}$. Two vertices of the same Q_i cannot be in different components, thus $\omega(G-S) \leq c_{\mathrm{a}} + 2t + 3$.

By the assumption, $\mathcal{R} \subseteq S$. If $a_i \in \mathcal{C}_{\mathrm{a}}$ then $N(a_i) \subseteq S$, so $q_i \in S$ for each i for which $a_i \in \mathcal{C}_{\mathrm{a}}$, and consequently $|S| \geq |X| + |\mathcal{R}| + c_{\mathrm{a}}$ holds.

From this inequality it follows by assumption that

$$(2t-\varepsilon)\left(c_{\mathrm{a}} + 2t + 3\right) > 2t + (2t-1)(2t+3)(2p+1) + c_{\mathrm{a}}$$

or, equivalently,

$$(2t-1-\varepsilon)\left(c_{\mathrm{a}} + 2t + 3\right) + 2t + 3 > 2t + (2t-1)(2t+3)(2p+1). \tag{4}$$

Using $2t-1-\varepsilon > 0$ and $c_{\mathrm{a}} \leq (2t+3)(2p+1)$ from (4) we obtain

$$(2t-1-\varepsilon)\,(2t+3)(2p+1) + (2t-\varepsilon)\,(2t+3) > 2t + (2t-1)(2t+3)(2p+1).$$

This is equivalent to

$$(2t-1-\varepsilon)(2p+1) + (2t-\varepsilon) > \frac{2t}{2t+3} + (2t-1)(2p+1).$$

This implies

$$\frac{t}{\varepsilon} > \frac{2t-\varepsilon-\frac{2t}{2t+3}}{2\varepsilon} - \frac{1}{2} > p$$

which is a contradiction since $p > \frac{2t}{\varepsilon}$. ■

This theorem shows that Theorem 5 is sharp for any integer t. It is still not known whether this is true for non–integer t values.

3. 1–edge–tough graphs have 2–factors

THEOREM B. *Let G be a 1–edge–tough graph on $n \geq 3$ vertices. Then G has a 2–factor.*

PROOF: By contraposition. Suppose that G does not have a 2–factor. It will be proved that G is not 1–edge–tough.

If G does not have a 2–factor then, by Theorem 7, there exists a subset of vertices U such that

$$|U| < \frac{1}{2}\left(|K_U| + \mu_U\right) \tag{5}$$

where K_U is the set of vertices having degree at most one in $G-U$ and where μ_U denotes the number of components of $G-U$ containing an odd number of vertices in K_U.

Let Z be the set of isolated vertices in $G-U$. The set of vertices with degree one in $G-U$ is denoted by A', i.e., $K_U = Z \cup A'$. Let $e_1, e_2, \ldots, e_f$ be the edges of $G-U$ with both endvertices in A'. These edges are not adjacent to any other edge in $G-U$. The set of endvertices of these edges is denoted by F and $A := A' - F$. Let $C_1, C_2, \ldots, C_c$, $(D_1, D_2, \ldots, D_d)$ denote the components of $G-U-Z-F$ containing an even (odd) number of vertices of A.

Let p_i denote the number of those vertices in C_i which are adjacent to at least two vertices of A and r_i the number of those vertices in C_i which are adjacent to exactly one vertex of A. Similarly, q_i and s_i are defined in D_i.

The assumption (5) can be reformulated with these notations as

$$|A| + |Z| + 2f + d + |Z| > 2|U|$$

or, equivalently,

$$\frac{|A|}{2} + |Z| + f > |U| - \frac{d}{2}. \tag{6}$$

Now we give a lower bound on $|A|$. It is easy to see that C_i has at least $2p_i + r_i$ vertices in A. If r_i is odd then this lower bound is odd as well. Since C_i must have an even number of vertices in A, $2p_i + r_i + 1$ is a better bound if r_i is odd. Thus the number of vertices of A in C_i is at least

$$2p_i + r_i + 2\left(\frac{r_i}{2} - \left\lfloor \frac{r_i}{2} \right\rfloor\right) = 2p_i + 2r_i - 2\left\lfloor \frac{r_i}{2} \right\rfloor \tag{7}$$

where $\lfloor x \rfloor$ denotes the largest integer which is not larger than x. Similarly the number of vertices of A in D_i is at least

$$2q_i + s_i + 2\left(\frac{s_i+1}{2} - \left\lfloor \frac{s_i+1}{2} \right\rfloor\right) = 2q_i + 2s_i + 1 - 2\left\lfloor \frac{s_i+1}{2} \right\rfloor. \tag{8}$$

Summing (7) and (8) for all components C_i and D_i,

$$|A| \geq 2\sum_1^c p_i + 2\sum_1^c r_i - 2\sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + 2\sum_1^d q_i + 2\sum_1^d s_i + d - 2\sum_1^d \left\lfloor \frac{s_i+1}{2} \right\rfloor$$

or equivalently

$$\frac{|A|}{2} \geq \sum_1^c p_i + \sum_1^c r_i - \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d q_i + \sum_1^d s_i + \frac{d}{2} - \sum_1^d \left\lfloor \frac{s_i+1}{2} \right\rfloor \tag{9}$$

is obtained.

The vertex set X and the edge set Y are defined next which will prove that the graph is not 1–edge–tough.

Let P denote the subset of those vertices of $G - U - Z - F$ whose vertices are adjacent to at least two vertices in A and Q the subset of vertices of $G - U - Z - F$ whose vertices are adjacent to exactly one vertex in A. Thus $|P| = \sum_1^c p_i + \sum_1^d q_i$ and $|Q| = \sum_1^c r_i + \sum_1^d s_i$. Let $X = U \cup P$.

Y is defined as follows. An edge of C_i or D_i belongs to Y if and only if it is incident neither to X nor to A. Formally $Y := E(G - X - A - Z - F)$. The components of the edge induced subgraph of $G(Y)$ are the components of $C_i - X$ and $D_i - X$ and the boundaries of these components are those vertices of C_i and D_i which are adjacent to exactly one vertex of A. The interiors of the components consist of those vertices which are not adjacent to A.

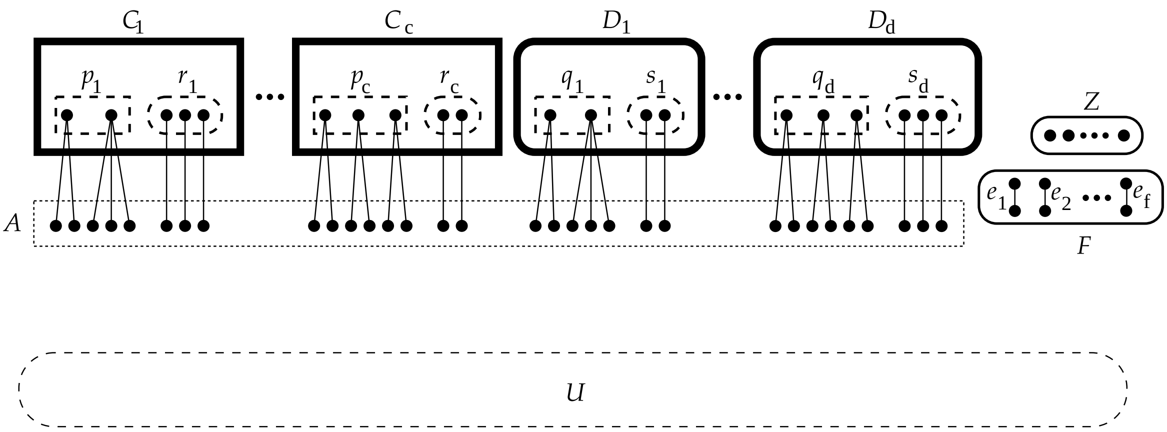

Figure 2

It is easy to see from Figure 2. that $G - X - Y - \cup_{i=1}^n \mathrm{in}_{G-X}(Q_i)$ is the union of the set of isolated vertices consisting of Z and those vertices of A which are adjacent to P and the set of independent edges $e_1, \ldots, e_f$ and the edges which connect A and Q. To finish the proof it remains to prove that these sets X and Y violate the inequality of Definition 1 for $t = 1$, namely the following inequality:

$$\omega\left(G - X - Y - \bigcup_{i=1}^n \mathrm{in}_{G-X}(Q_i)\right) = |A| + |Z| + f > |X| + \sum_{i=1}^n \left\lfloor \frac{|\mathrm{bd}_{G-X}(Q_i)|}{2} \right\rfloor = s(X, Y; G).$$

In our case $\sum_1^n \left\lfloor \frac{|\mathrm{bd}_{G-X}(Q_i)|}{2} \right\rfloor \le \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d \left\lfloor \frac{s_i}{2} \right\rfloor$. Note that equality does not necessary hold because Q may have more than one component in any C_i or D_i. Thus for deriving the desired contradiction it is enough to prove

$$|A| + |Z| + f > |U| + \sum_1^c p_i + \sum_1^d q_i + \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d \left\lfloor \frac{s_i}{2} \right\rfloor . \tag{10}$$

Summing (6) and (9) we obtain

$$|A| + |Z| + f > |U| - \frac{d}{2} + \sum_1^c p_i + \sum_1^c r_i - \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d q_i + \sum_1^d s_i + \frac{d}{2} - \sum_1^d \left\lfloor \frac{s_i+1}{2} \right\rfloor . \tag{11}$$

Since $x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor$ and $x \ge 2 \left\lfloor \frac{x}{2} \right\rfloor$ for any non negative integer x it is easy to see that

$$\sum_1^c r_i - \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d s_i - \sum_1^d \left\lfloor \frac{s_i+1}{2} \right\rfloor \ge \sum_1^c \left\lfloor \frac{r_i}{2} \right\rfloor + \sum_1^d \left\lfloor \frac{s_i}{2} \right\rfloor$$

holds. Thus (10) can be obtained from (11), completing the proof. ■

It should be mentioned that the special case of the above construction for $t = 1$ is very similar to a construction of Bauer, Broersma, van den Heuvel and Veldman [1].

It is natural to make the following conjecture.

CONJECTURE. *Let G be a t–edge–tough graph on $n \ge 2t+1$ vertices where t is a positive integer. Then G has a $2t$–factor.*

If this were true, it would be a sharpening of Theorem 2.

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