Chordality and 2-Factors in Tough Graphs

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Abstract

A graph G is *chordal* if it contains no chordless cycle of length at least four and is *k-chordal* if a longest chordless cycle in G has length at most k. In this note it is proved that all $\frac{3}{2}$-tough 5-chordal graphs have a 2-factor. This result is best possible in two ways. Examples due to Chvátal show that for all $\epsilon > 0$ there exists a $(\frac{3}{2} - \epsilon)$-tough chordal graph with no 2-factor. Furthermore, examples due to Bauer and Schmeichel show that the result is false for 6-chordal graphs.

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1 Introduction

We begin with a few definitions and some notation. Other definitions will be given later, as needed. A good reference for any undefined terms is [7]. We consider only undirected graphs with no loops or multiple edges. Let G be a graph. Then G is *hamiltonian* if it has a *Hamilton* cycle, i.e., a cycle containing all of its vertices. It is *traceable* if it has a path containing all of its vertices. Let $\omega(G)$ denote the number of components of G. Then G is *t-tough* if $|S| \geq t \cdot \omega(G-S)$ for every subset S of the vertex set V of G with $\omega(G-S) > 1$. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = \frac{(n-1)}{2}$ for all $n \geq 1$). A *k-factor* is a k-regular spanning subgraph. Of course, a Hamilton cycle is a 2-factor. We say G is *chordal* if it contains no chordless cycle of length at least four and is *k-chordal* if a longest chordless cycle in G has length at most k.

Our work was motivated by a desire to understand the relationship between the toughness of a graph and its cycle structure. For a survey of recent work in this area, see [3, 4, 5]. Toughness was introduced by Chvátal in [9]. An obvious connection between toughness and hamiltonicity is that being 1-tough is a necessary condition for a graph to be hamiltonian. Chvátal conjectured that there exists a finite constant t_0 such that every t_0-tough graph is hamiltonian. This conjecture is still open. Until recently it was believed that the smallest value of t_0 for which this might be true was $t_0 = 2$. We now know this is false.

Theorem 1.1 [1]. *For every $\epsilon > 0$, there exists a $(\frac{9}{4} - \epsilon)$-tough nontraceable graph.*

Chvátal also conjectured that every k-tough graph on n vertices with $n \geq k+1$ and kn even has a k-factor. This was established in [10].

Theorem 1.2 [10]. *Let G be a k-tough graph on n vertices with $n \geq k+1$ and kn even. Then G has a k-factor.*

It was also shown in [10] that Theorem 1.2 is best possible.

Theorem 1.3 [10]. *Let $k \geq 1$. For any $\epsilon > 0$, there exists a $(k - \epsilon)$-tough graph G on n vertices with $n \geq k+1$ and kn even which has no k-factor.*

The above results imply that while 2-tough graphs have 2-factors, there exists an infinite sequence of graphs without 2-factors having toughness approaching 2. In [11] it was shown that a similar statement holds for split graphs. A graph G is called a *split graph* if its vertices can be partitioned into an independent set and a clique.

Theorem 1.4 [11]. *Every $\frac{3}{2}$-tough split graph is hamiltonian.*

In [[9], p.223], Chvátal found a sequence $\{G_n\}_{n=1}^{\infty}$ of non-2-factorable graphs with $\tau(G_n) \to \frac{3}{2}$. These graphs were in fact split graphs.

Theorem 1.5 *There is a sequence* $\{G_n\}_{n=1}^{\infty}$ *of non-2-factorable split graphs with* $\tau(G_n) \to \frac{3}{2}$.

In this note we prove that all $\frac{3}{2}$-tough chordal graphs have a 2-factor. In fact we prove a bit more.

Theorem 1.6 *Let G be a $\frac{3}{2}$-tough 5-chordal graph. Then G has a 2-factor.*

Since all split graphs are chordal, the graphs Chvátal constructed in [9] are also chordal. Thus Theorem 1.6 is best possible with respect to toughness. Furthermore, the graphs $G_{l,m}$ in [[2], p.251] are 6-chordal graphs without a 2-factor. By choosing l and m large the toughness of these graphs can be made to approach 2 from below. Note that Theorem 1.6 is in some sense the definitive result of the form "If G is a t-tough k-chordal graph, then G has a 2-factor": it follows from the examples in [9] that this is false for $t < \frac{3}{2}$ and any k, by Theorem 1.2 it is true for $t \geq 2$ and any k, and from the examples in [2] it follows that for $\frac{3}{2} \leq t < 2$ the best one can hope for is a result with $k = 5$.

Unlike the case with split graphs, however, it is not true that all $\frac{3}{2}$-tough chordal graphs are hamiltonian.

Theorem 1.7 [1]. *For every $\epsilon > 0$ there exists a $(\frac{7}{4} - \epsilon)$-tough chordal nontraceable graph.*

Recently, Chen, Jacobson, Kézdy and Lehel [8] have shown that every 18-tough chordal graph is hamiltonian. We now conjecture the following.

Conjecture: Every 2-tough chordal graph is hamiltonian and for every $\epsilon > 0$ there exists a $(2 - \epsilon)$-tough chordal nonhamiltonian graph.

Returning to 2-factors, it is natural to ask how large the minimum vertex degree of a t-tough $(1 \leq t < 2)$ graph can be, if the graph contains no 2-factor. This problem was answered in [2] for $1 \leq t \leq \frac{3}{2}$ and for infinitely many t satisfying $\frac{3}{2} \leq t < 2$. A key lemma (Lemma 8) in [2] is the basis for the proof of our main result. Of course, any paper dealing with sufficient conditions for a graph to have a regular factor relies heavily on a well-known theorem of Belck [6] and Tutte [12]. This result is given in Section 2. The proof of our main result appears in Section 3.

2 Preliminary Results

Let G be a graph. If A and B are subsets of V or subgraphs of G, and $v \in V$, we use $e(v, B)$ to denote the number of edges joining v to a vertex of B, and $e(A, B)$ to denote $\sum_{v \in A} e(v, B)$. We use $\langle A \rangle$ to denote the subgraph of G induced by A. A vertex $v \in V$ will be called *complete* if v is adjacent to every other vertex in V, and is called *simplicial* if the subgraph induced by the neighborhood of v is complete.

Our proof of Theorem 1.6 relies heavily on a theorem that characterizes those graphs not containing a 2-factor. This theorem is a special case of the theorems of Belck [6] and Tutte [12]. For disjoint subsets A, B of $V(G)$ let odd (A, B) denote the number of components H of $G - (A \cup B)$ with $e(H, B)$ odd, and let

$$\Theta(A, B) = 2|A| + \sum_{y \in B} d_{G-A}(y) - 2|B| - \text{ odd } (A, B).$$

Theorem 2.1 [6], [12].
Let G be any graph. Then

(i) for any disjoint sets $A, B \subseteq V(G), \Theta(A, B)$ is even;

(ii) the graph G does not contain a 2-factor if and only if $\Theta(A, B) \leq -2$ for some disjoint pair of sets $A, B \subseteq V(G)$.

We call a pair (A, B) of disjoint subsets of $V(G)$ with $\Theta(A, B) \leq -2$ a *Tutte pair* for G. Note that in any Tutte pair (A, B) for G we have $B \neq \emptyset$, since by definition $\sum_{y \in B} d_{G-A}(y) \geq$ odd (A, B) and so $\Theta(A, B) \leq -2$ implies $|B| > |A| \geq 0$. We define a Tutte pair (A, B) to be *minimal* if $\Theta(A, B') \geq 0$ for any proper subset $B' \subseteq B$. Clearly any graph without a 2-factor contains a minimal Tutte pair.

The next lemma follows easily from a result in [10]. The proof also appears in [2].

Lemma 2.2 *Let G be a graph having no 2-factor. If (A, B) is a minimal Tutte pair for G, then B is an independent set.*

To facilitate the proof in the next section we define a Tutte pair (A, B) to be a *strong Tutte pair* if B is an independent set.

3 Proof of Theorem 1.6

We begin with the following lemma, which is also implicit in [2].

Lemma 3.1 *Let v be a simplicial vertex in a non-complete graph G. Then $\tau(G - v) \geq \tau(G)$.*

Proof: First denote $G - v$ by G_v. Note that if G_v is complete, then $\tau(G_v) = \frac{|V(G_v)| - 1}{2} = \frac{|V(G)| - 2}{2} \geq \tau(G)$. Suppose $\tau(G_v) < \tau(G)$. Then there exists $X \subseteq V(G_v)$ such that $\omega(G_v - X) \geq 2$ and $\frac{|X|}{\omega(G_v - X)} < \tau(G)$. However $\omega(G - X) \geq \omega(G_v - X) \geq 2$, since the neighbors of v in G induce a complete subgraph. But this gives $\frac{|X|}{\omega(G - X)} \leq \frac{|X|}{\omega(G_v - X)} < \tau(G)$, a contradiction. □

Proof of Theorem 1.6:

Let G be a $\frac{3}{2}$-tough 5-chordal graph having no 2-factor and (A, B) be a strong Tutte pair for G, existing by Lemma 2.2. Thus $\Theta(A, B) \leq -2$. Let $C = V(G) - (A \cup B)$. Since B is an independent set of vertices, $\sum_{y \in B} d_{G-A}(y) = e(B, C)$. Hence by Theorem 2.1,

$$2|A| + e(B, C) \leq 2|B| + \text{ odd } (A, B) - 2. \tag{1}$$

Among all possible choices, we choose G and the strong Tutte pair (A, B) as follows:

(i) $|V(G)|$ is minimal;

(ii) $|E(G)|$ is maximal, subject to (i);

(iii) $|B|$ is minimal, subject to (i) and (ii);

(iv) $|A|$ is maximal, subject to (i), (ii) and (iii).

We now show that G has properties (a)-(g) below.

(a) For any $x \in B$ and any component H of $\langle C \rangle, e(x, H) \leq 1$.

Proof of (a):

Let $x \in B$ with $d_{G-A}(x) = k$, and let $C_1, C_2, \ldots, C_j$ denote the components of $\langle C \rangle$ to which x is adjacent. If $j \leq k - 1$, delete x from B and add x to C (thus redefining B and C). Since odd (A, B) has decreased by at most $j \leq k - 1$, it is easy to check that $\Theta(A, B)$ has increased by at most 1. Thus we still have $\Theta(A, B) \leq -2$ (by Theorem 2.1(i)) and we contradict (iii).

(b) The vertices of A are complete.

Proof of (b):

If not, form a new graph G' by adding the edges required to make the vertices of A complete. Clearly G' is still $\frac{3}{2}$-tough and (A, B) is still a strong Tutte pair for G'. Obviously, no chordless cycle of G' can contain a vertex of A. Since G is 5-chordal, it follows that G' is also 5-chordal. Thus we contradict (ii).

(c) For any $y \in C, e(y, B) \leq 1$.

Proof of (c):

Suppose that $e(y, B) \geq 2$ for some $y \in C$. Delete y from C and add y to A (thus redefining A and C). It is easy to check that (A, B) remains a strong Tutte pair. Thus we contradict (iv).

(d) Each component of $\langle C \rangle$ is a complete graph.

Proof of (d):

If not, form a new graph G' by adding the edges required to make each component $C_1, C_2, \ldots, C_s$ of $\langle C \rangle$ a complete graph. Clearly, G' is still $\frac{3}{2}$-tough and (A, B) is still a strong Tutte pair for G'. Assuming G' is not 5-chordal, let C^* be a shortest chordless cycle in G' of length at least 6. Clearly C^* can not contain a vertex of A, nor can it have more than two vertices from any component of $\langle C \rangle$. Since B is independent, C^* is of the form

$$C^* : b_1 T_1' b_2 T_2' \ldots b_k T_k' b_1,$$

where, for $1 \leq i \leq k$, each T_i' represents an edge $t_i^1 t_i^2$ of a component C_i in G'.

Form the cycle C^{**} in G by taking C^* and substituting T_i for $T_i' (1 \leq i \leq k)$, where T_i is a shortest $t_i^1 - t_i^2$ path in C_i in G. The graph G is 5-chordal, so C^{**} has a chord. Since any chord of C^{**} must join a vertex of B and a vertex of C and C^* is a chordless cycle in G', we may assume, without loss of generality, that there exists a chord $b_1 u$ of C^{**} such that

- u is an internal vertex of some T_i, say of T_m, and
- the cycle $b_1 T_1 b_2 T_2 \ldots b_m U b_1$, where U is the $t_m^1 - u$ subpath of T_m, is chordless.

By (a) we have $1 < m < k$. But then $b_1 T_1' b_2 T_2' \ldots b_m t_m^1 u b_1$ is a chordless cycle in G' of length at least 6 which is shorter than C^*, contradicting the choice of C^*. Thus G' is 5-chordal and we contradict (ii).

(e) For any $y \in C, e(y,B) = 1$ (and thus $e(B,C) = |C|$).

Proof of (e):

Suppose now that C contains a vertex y with $e(y,B) = 0$. It follows from (b) and (d) that v is simplicial. Hence by Lemma 3.1 , $\tau(G-y) \geq \tau(G)$. Furthermore, (A,B) is still a strong Tutte pair for the 5-chordal graph $G-y$. Hence, by (i), the graph $G-y$ contradicts the choice of G.

(f) $|B| \geq 2$.

Proof of (f):

We saw earlier that $|B| > |A| \geq 0$, and so $|B| \geq 1$. Suppose $B = \{x\}$. Since (A,B) is a Tutte pair with $|B| = 1$ and $|A| = 0$, we have $e(B,C) \leq$ odd (A,B) by (1). If $e(B,C) \geq 2$, then $\omega(G-B) \geq$ odd $(A,B) \geq e(B,C) \geq 2 > |B|$, and G is not 1-tough. If $e(B,C) = 1$, then G is not 1-tough either. Hence $|B| \geq 2$.

(g) odd $(A,B) = \omega(\langle C \rangle)$.

Proof of (g):

Suppose there exists a component C_i in $\langle C \rangle$ with $e(C_i, B) = |C_i|$, an even integer. Let y be any vertex in C_i. Add y to A, thus redefining A and C. It is easy to see that (A,B) is still a strong Tutte pair for G. Thus we contradict (iv).

Hence G and its minimal Tutte pair (A,B) have properties (a) - (g). Set $s = \omega(\langle C \rangle) =$ odd (A,B).

Consider the components $C_1, C_2, \ldots, C_s$ of $\langle C \rangle$ and let $y_j \in V(C_j)$. Define $X = A \cup C - \{y_1, \ldots, y_s\}$. Since B is independent and $e(y_i, B) = 1$ for $1 \leq i \leq s$, we have $\omega(G-X) = |B| \geq 2$. For convenience let $a = |A|, b = |B|$ and $c = |C|$. Using properties (e), (g) and inequality (1), we have

$$\frac{3}{2} \leq \frac{|X|}{\omega(G-X)} = \frac{a+c-s}{b} = \frac{a+e(B,C)-\mathrm{odd}(A,B)}{b} \leq \frac{2b-a-2}{b}.$$

Hence

$$b \geq 2a+4. \tag{2}$$

To complete the proof we establish the following.

Claim: $b \geq c - s + 1$.

Once the claim is established, it follows that

$$\frac{3}{2} \leq \frac{|X|}{\omega(G-X)} = \frac{a+c-s}{b} \leq \frac{a+b-1}{b}.$$

Thus

$$b \leq 2a - 2. \tag{3}$$

The fact that (2) and (3) are contradictory completes the argument.

Proof of Claim:

Form a bipartite graph F from G by deleting A and contracting each component of $\langle C \rangle$ into a single vertex. By (a), F has no multiple edges. The key observation is that since G is 5-chordal, F is a forest. Otherwise, let C_F be a shortest cycle in F. Then C_F is of the form

$$C_F : b_1 T_1 b_2 T_2 \ldots b_p T_p b_1,$$

where each $T_i, 1 \leq i \leq p$, represents the contracted component C_i. By (d) and (e), it follows that the 2 edges incident with each T_i in C_F correspond to edges $b_i t_i^1, b_{i+1} t_i^2$, where $t_i^1 t_i^2$ is an edge in C_i. It follows that G has a chordless cycle of length at least 6, a contradiction.

Hence

$$\sum_{v \in C} d_F(v) = c = |E(F)| \leq |V(F)| - 1 = b + s - 1.$$

Thus $b + s - 1 \geq c$ and the claim is established. $\square$

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