Hamiltoninan Chains in Hypergraphs
A survey

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Abstract

Hamiltionian chain is a generalization of hamiltonian cycles for hypergraphs. Among the several possible ways of generalizations this is probably the most strong one, it requires the strongest structure. Since there are many interesting questions about hamiltonian cycles in graphs, we can try to answer these questions for hypergraphs, too. In the present article we give a survey on results about such questions.

AMS subject classification (2000): 05C45, 05C65

1 Introduction

Let $\mathcal{H}$ be a r-uniform hypergraph on the vertex set $V(\mathcal{H}) = \{v_1, v_2, \ldots, v_n\}$ where $n > r$. For simplicity of notation v_{n+x} with $x \geq 0$ denotes the same vertex as v_x (unless stated otherwise). The set of the edges, r-element subsets of V, is denoted by $\mathcal{E}(\mathcal{H}) = \{E_1, E_2, \ldots, E_m\}$. We will write simply V for $V(\mathcal{H})$ and $\mathcal{E}$ for $\mathcal{E}(\mathcal{H})$ if no confusion can arise.

In [9] the authors defined the notion of a hamiltonian-chain.

Definition 1 *A cyclic ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set is called a* Hamiltionian chain *iff for each $1 \leq i \leq n$ $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\} =: E_j$ is an edge of $\mathcal{H}$. An ordering $(v_1, v_2, \ldots, v_{l+1})$ of a subset of the vertex set is called an* open chain of length l *between v_1 and v_{l+1} iff for each $1 \leq i \leq l - r + 2$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\} = E_j$. An open chain of length $n-1$ is an* open hamiltonian chain. *A cyclic ordering $(v_1, v_2, \ldots, v_l)$ of a subset of the vertex set is called a* chain of length l *iff for every $1 \leq i \leq l$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\} = E_j$. (Now v_{l+x} denotes the same vertex as v_x).*

Since an ordinary graph is a 2-uniform hypergraph, this definition gives the definition of the hamiltonian cycle in ordinary graphs for $r = 2$.

Many questions about hamiltonian cycles involve some degree condition on the graph. The notion of the degree is also extended for hypergraphs, it is defined below in full generality, however, only some special cases will be used.

*Research supported by OTKA grants OTKA T 046234 and T 043520

Definition 2 *The* degree of a fixed l-tuple *of distinct vertices,* $\{v_1, v_2, \ldots, v_l\}$*, in a* r*-uniform hypergraph is the number of edges of the hypergraph containing the set* $\{v_1, v_2, \ldots, v_l\}$*. It is denoted by* $d_r(v_1, v_2, \ldots, v_l)$*. Furthermore* $\delta_r^{(l)}(\mathcal{H})$ *denotes the minimum of* $d_r(v_1, v_2, \ldots, v_l)$ *over all* l*-tuples of vertices in* $\mathcal{H}$*. The* neighborhood *of a vertex* v *is defined by*

$$N_{\mathcal{H}}(v) := \{E - \{v\} \mid v \in E, E \in \mathcal{E}(\mathcal{H})\}.$$

Questions about hamiltonian cycles are usually not easy. One reason of this is probably that the problem of deciding whether a given graph contains a hamiltonian cycle is an **NP**-complete problem. From this point of view we cannot expect the hypergraph versions of these questions to be easier, since the corresponding problem for Hamiltionian chain is also **NP**-complete for any fixed r. This was proved by Heiko Müller [11].

An other reason might be the following. In graphs, if there is a path from $v \in V(G)$ to $u \in V(G)$ and there is a path from $u \in V(G)$ to $w \in V(G)$, then there is a path from $v \in V(G)$ to $w \in V(G)$. This fact makes it possible to define components in graphs. However, the above claim does not hold in hypergraphs, so it is more complicated to find a good notion replacing "components".

In each of the following four Sections we will deal with a separate question about hamiltonian chains.

2 Dirac type theorem

Dirac [5] proved that if the minimal degree of a graph on n vertices is at least $\frac{n}{2}$ then it has a hamiltonian cycle (chain). This is tight, as it is demonstrated by the graph obtained from two disjoint nearly equal cliques and another vertex that is adjacent to every vertex in the union of the cliques. In [9] the authors attempt to generalize this result to hamiltonian chains in r-uniform hypergraphs $\mathcal{H}$ with large l-minimum degree.

Theorem 3 ([9]) *If* $\mathcal{H} = (V, \mathcal{E})$ *is an* r*-uniform hypergraph on* n *vertices with* $\delta_{r-1}(\mathcal{H}) > (1 - \frac{1}{2r})n + 4 - r - \frac{5}{2r}$*, then* $\mathcal{H}$ *contains a hamiltonian chain.*

The bound given in Theorem 3 is probably not best possible. However, it is quite easy to show that this bound cannot be lower than $\left\lfloor \frac{n-r+1}{2} \right\rfloor$.

Theorem 4 ([9]) *For all integers* r *and* n *such that* $2 \le r$ *and* $r^2 < n$*, there exists a hypergraph* $\mathcal{H} = (V, \mathcal{E})$ *on* n *vertices which does not contain a hamiltonian chain such that*

$$\delta_{r-1}(\mathcal{H}) \ge \left\lfloor \frac{n-r+1}{2} \right\rfloor.$$

PROOF: First we construct a graph $\mathcal{H}' = (V', \mathcal{E}')$ on $n-1$ vertices with $\delta_{r-1}(\mathcal{H}') \ge \left\lfloor \frac{n-r-1}{2} \right\rfloor$ such that $\mathcal{H}'$ does not have an open hamiltonian chain. Then $\mathcal{H}$ will be the result of adding a new vertex v to V' and all r-sets containing v to $\mathcal{E}'$.

Partition V into sets $X = \left\{x_1, \ldots, x_{\lfloor \frac{n-1}{2} \rfloor}\right\}$ and $Y = \left\{y_1, \ldots, y_{\lceil \frac{n-1}{2} \rceil}\right\}$. Let $\mathcal{E}$ consist of all r-subsets $E \subset V$ such that $|E \cap X| \ne \left\lfloor \frac{r-1}{2} \right\rfloor$.

Suppose that $(v_1, \ldots, v_{n-1})$ is an open hamiltonian chain in $\mathcal{H}$. Let $E_i = \{v_i, \ldots, v_{i+r-1}\}$ and $\xi_i = |E_i \cap X|$. It is easy to see that $|\xi_{i+1} - \xi_i| \le 1$. Consider the following inequalities.

$$r\left\lfloor \frac{n-1}{2} \right\rfloor - \binom{r}{2} \le \sum_{i=1}^{n-r} \xi_i \le r\left\lfloor \frac{n-1}{2} \right\rfloor$$

The upper bound holds because each vertex of X is counted once for each edge E_i containing it and each vertex is contained in at most r edges E_i. The lower bound holds because every vertex of X is counted at least once and only the last $i \le r$ vertices of X can be counted at most i times. Since $\xi_i \neq \lfloor \frac{r-1}{2} \rfloor$ for $1 \le i \le n-r$, either $\xi_i \le \lfloor \frac{r}{2} \rfloor - 1$ for all $1 \le i \le n-r$ or $\xi_i \ge \lfloor \frac{r}{2} \rfloor + 1$ for all $1 \le i \le n-r$. In the first case

$$r\frac{n-2}{2} - \binom{r}{2} \le r\left\lfloor \frac{n-1}{2} \right\rfloor - \binom{r}{2} \le \sum_{i=1}^{n-r} \xi_i \le (n-r)\left\lfloor \frac{r-2}{2} \right\rfloor \le (n-r)\frac{r-2}{2}$$

This is a contradiction since $n > \frac{3}{2}r$. In the second case

$$(n-r)\frac{r+1}{2} \le (n-r)\left\lfloor \frac{r+2}{2} \right\rfloor \le \sum_{i=1}^{n-r} \xi_i \le r\left\lfloor \frac{n-1}{2} \right\rfloor \le r\frac{n-1}{2}.$$

This is a contradiction since $n > r^2$.

Now consider the degree condition. Let S be any $(r-1)$ -subset of V. First suppose that $|S \cap X| = \lfloor \frac{r}{2} \rfloor$. Since adding any vertex form $X \backslash S$ to S yields an edge of the hypergraph, $d(S) \ge \lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{r}{2} \rfloor$. Next suppose that $|S \cap X| = \lfloor \frac{r}{2} \rfloor - 1$. Since adding any vertex from $Y \backslash S$ to S yields an edge of the hypergraph, $d(S) \ge \lceil \frac{n-1}{2} \rceil - \lceil \frac{r}{2} \rceil$. Finally suppose that $|S \cap X| \notin \left\{ \lfloor \frac{r}{2} \rfloor, \lfloor \frac{r}{2} \rfloor - 1 \right\}$. Since adding any vertex from $V \backslash S$ to S yields an edge, $d(S) \ge n - r$. Therefore

$$\delta_{r-1}(\mathcal{H}') \ge \min\left(\left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{r}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil - \left\lceil \frac{r}{2} \right\rceil \right) = \left\lfloor \frac{n-r-1}{2} \right\rfloor.$$

To see that $\mathcal{H}$ cannot be extended, consider a non-edge, say

$$E = \left\{ x_1, \ldots, x_{\lfloor \frac{r}{2} \rfloor}, y_1, \ldots, y_{\lceil \frac{r}{2} \rceil} \right\} \notin \mathcal{E}.$$

Then $\left(v, x_{\lfloor \frac{n-1}{2} \rfloor}, \ldots, x_1, y_1, \ldots, y_{\lceil \frac{n-1}{2} \rceil} \right)$ is a hamiltonian chain in $\mathcal{H} + E$. $\square$

A few years later Rödl, Ruciński and Szemerédi proved that this bound is tight if $r = 3$ and n is sufficiently large. Since the proof uses the hypergraph version of Szemerédi's regularity lemma, sufficiently large here means extremely large.

Theorem 5 ([14]) *Let $\mathcal{H} = (V, \mathcal{E})$ be an 3-uniform hypergraph on n vertices with $\delta_2(\mathcal{H}) \ge \lfloor \frac{n-1}{2} \rfloor$, where n is large enough. Then $\mathcal{H}$ contains a hamiltonian chain.*

Recently the same authors announced that they can prove the above result for general r and also they improved the lower bound on n, but is still huge. Therefore this question is seems to be more or less settled, but it would be still nice to have a proof for "normal size" hypergraphs, too.

3 k-edge-hamiltonian hypergraphs

Definition 6 *A hypergraph is* hamiltonian *if it contains a hamiltonian-chain and it is* k-edge-hamiltonian *if by the removal of any k edges a hamiltonian hypergraph is obtained.*

The main aim of the present section is to investigate minimum size k-edge-hamiltonian hypergraphs. In [13, 16] Paoli, Wong and Wong settled this question for graphs.

Theorem 7 ([13, 16]) *The number of edges in a minimum k-edge-hamiltonian graph on $n \ge k+3$ vertices is $\lceil n(k+2)/2 \rceil$.*

Since the degree of any vertex in an r-uniform hamiltonian chain is r, the minimum degree in a k-edge-hamiltonian hypergraph is at least $r+k$, so the number of edges is at least $\lceil n(r+k)/r \rceil$. For $r=2$ this shows that the constructions in the above theorem are best possible. However, for $r>2$ this lower bound is not best possible.

In [8] the authors establishes some upper and lower bounds for some special cases of r and k.

If a hypergraph contains $k+1$ edge-disjoint hamiltonian chains, then it is clearly k-edge-hamiltonian. This observation leads to the trivial upper bound on the minimum number of edges: $(k+1)n$. If $k=1$ and $r=3$ then the following slightly better upper bound is obtained.

Theorem 8 ([8]) *There exists a 1-edge-hamiltonian 3-uniform hypergraph $\mathcal{H}$ on n vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{11}{6}n + o(n).$$

Proof: Let $\mathcal{V}(\mathcal{H}) := \{w_1, \dots, w_p, v_1, \dots, v_q\}$ where $p = \lceil n/6 \rceil$ and $q = n - p$. There are two types of edges in $\mathcal{H}$. The first kind of edges form a chain on $\{v_1, \dots, v_q\}$,

$$\mathcal{E}_1(\mathcal{H}) := \left\{ \{v_i, v_{i+1}, v_{i+2}\} \;\middle|\; 1 \le i \le q \right\}.$$

The second kind connects the rest of the vertices to this chain:

$$\mathcal{E}_2(\mathcal{H}) := \left\{ \{w_i, v_{5(i-1)+j}, v_{5(i-1)+j+1}\} \;\middle|\; 1 \le i \le p, 1 \le j \le 6 \right\}.$$

This means that the neighborhood of w_i is an ordinary graph, a path of length 6 formed by vertices $v_{5(i-1)+1}, \dots, v_{5(i-1)+7}$. The neighborhood of w_{i+1} is also a path of length 6, which begins at $v_{5(i-1)+6}$, so $v_{5(i-1)+6}, v_{5(i-1)+7} \in N(w_i) \cap N(w_{i+1})$ (except maybe for $N(w_1)$ and $N(w_p)$ where the overlap is larger if $6 \nmid n$). Let $\mathcal{E}(\mathcal{H}) := \mathcal{E}_1(\mathcal{H}) \cup \mathcal{E}_2(\mathcal{H})$, then it is clear that $|\mathcal{E}(\mathcal{H})| = q + 6p = n + 5\lceil n/6 \rceil = 11n/6 + o(n)$. (See Fig. 1.)

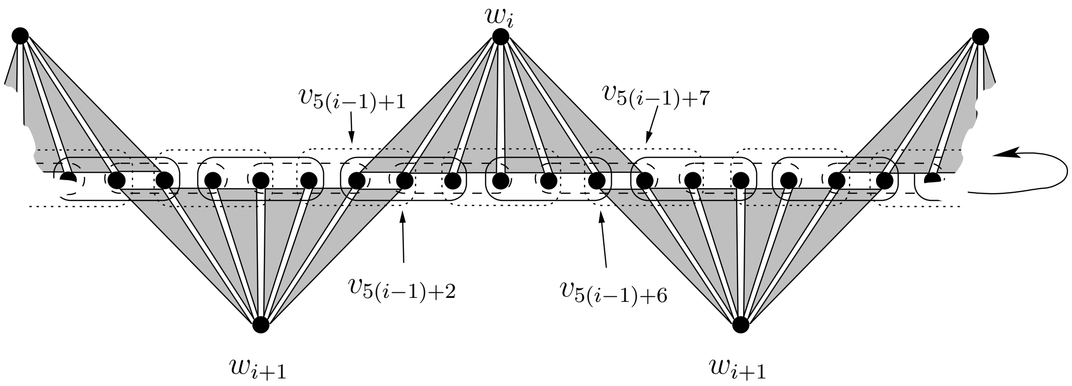

Figure 1: 3-uniform 1-edge-hamiltonian hypergraph

This hypergraph contains many hamiltonian-chains which can be obtained in the following way. Start with the chain formed by $\{v_1, \dots, v_q\}$ and extend this cycle by inserting the rest of the vertices one by one. It is obvious that we can insert w_i between any two consecutive vertices of $v_{5(i-1)+2}, v_{5(i-1)+3}, v_{5(i-1)+4}, v_{5(i-1)+5}, v_{5(i-1)+6}$ (but we cannot insert it between $v_{5(i-1)+1}$ and $v_{5(i-1)+2}$ or $v_{5(i-1)+6}$ and $v_{5(i-1)+7}$). Note that the new chain contains 3 "consecutive edges" of $N(w_i)$ but it does not contain 2 "consecutive edges" from the original chain (those which contain both neighbors of w_i in the new chain (See Fig. 2).

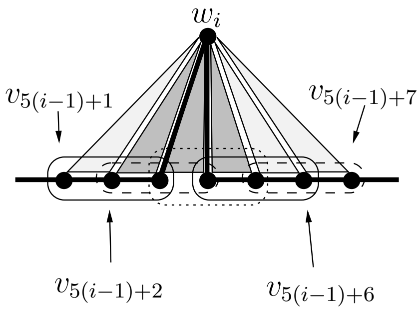

Figure 2: How to insert w_i?

Now we prove that $\mathcal{H}$ is 1-edge-hamiltonian, that is, $\mathcal{H} - E$ contains a hamiltonian-chain for any $E \in \mathcal{E}(\mathcal{H})$.

Suppose that $E = \{v_t, v_{t+1}, v_{t+2}\} \in \mathcal{E}_1(\mathcal{H})$. Then it is easy to check that there is a w_i which we can insert either between v_t and v_{t+1} or v_{t+1} and v_{t+2}, so the new chain does not contain E any more. Further, we can insert all other w vertices into suitable places, hence we obtain the desired hamiltonian-chain (see Fig. 3), for example the following one

$$v_t, v_{t+1}, w_i, v_{t+2}, v_{t+3}, \ldots, v_{t+5}, v_{t+6}, w_{i+1}, v_{t+7}, v_{t+8}, \ldots, v_{t+5j}, v_{t+5j+1}, w_{i+j}, v_{t+5j+2}, \ldots$$

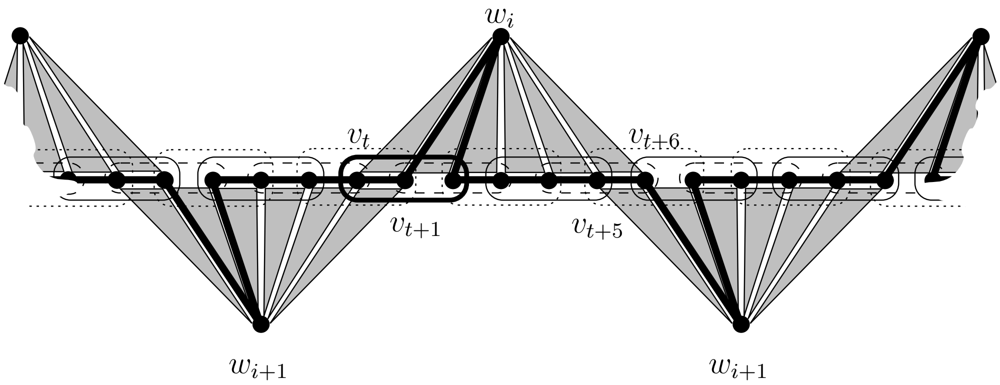

Figure 3: Hamiltonian chain in $\mathcal{H} - E$

On the other hand, if $w_i \in E$ for some i then it is clear that $N(w_i) - E$ always contains 3 "consecutive edges", therefore w_i can be inserted into the chain formed by $\{v_1, \ldots, v_q\}$. Inserting the rest of the vertices in the same way as in the other case, we obtain a hamiltonian-chain of $\mathcal{H} - E$. $\square$

Theorem 9 ([8]) *For any* 1*-edge-hamiltonian* 3*-uniform hypergraph* $\mathcal{H}$ *on* $n \geq 5$ *vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{14}{9} n$$

holds.

The proof uses some observations about the structure of the neighborhood of the vertices. It is important to note, that in a 3-uniform hypergraph the neighborhood of a vertex is not just a set of vertices, but a (2-uniform) graph. In general, the neighborhood of a vertex in an r-uniform hypergraph is an $(r-1)$-uniform hypergraph. The first important observation in the proof is that if there is a hamiltonian chain in a hypergraph, then the neighborhood of any vertex must contain an $(r-1)$-uniform chain of length $2r-3$.

Using similar ideas the authors prove some other special cases, too.

Theorem 10 ([8]) *There exists a 2-edge-hamiltonian 3-uniform hypergraph $\mathcal{H}$ on n vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{13}{4}n + o(n).$$

Theorem 11 ([8]) *For any 2-edge-hamiltonian 3-uniform hypergraph $\mathcal{H}$ on n vertices*

$$|\mathcal{E}(\mathcal{H})| \geq 2n.$$

holds.

The methods used in the proof of the theorem suggests the following conjecture.

Conjecture 12 ([8]) *For any k-edge-hamiltonian 3-uniform hypergraph $\mathcal{H}$ on n vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{S(k)}{3}n.$$

holds, where

$$S(k) = \left\lceil k + \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil + O(1).$$

Now we state lower and upper bounds for $r > 3$.

Theorem 13 ([8]) *There exists a 1-edge-hamiltonian r-uniform hypergraph $\mathcal{H}$ on n vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{4r-1}{2r}n + o(n).$$

Theorem 14 ([8]) *For any 1-edge-hamiltonian 4-uniform hypergraph $\mathcal{H}$ on $n \geq 6$ vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{3}{2}n$$

holds.

The question raised in this section is still open even for the spacial cases. It looks difficult to find the exact constant factor already for the $r = 3, k = 1$ case.

4 Maximally non-hamiltonian hypergraphs

What is the maximum number of edges in an r-uniform hypergraph on n vertices which has no hamiltonian chain?

Øre [12] proved that for graphs there is a unique maximal graph G with $\binom{n-1}{2} + 1$ edges. The graph G is formed from a $(n-1)$-clique together with one additional edge incident to the remaining vertex.

First, the authors in [9] gave an easy lower bound.

Theorem 15 ([9]) *If an r-uniform hypergraph $\mathcal{H}$ on n vertices has no hamiltonian chain then*

$$|\mathcal{E}(\mathcal{H})| < \left(1 - \frac{1}{n}\right)\binom{n}{r}$$

holds.

Later this was slightly improved in [8].

Theorem 16 ([8]) *If an r-uniform hypergraph $\mathcal{H}$ on n vertices has no hamiltonian chain then*

$$|\mathcal{E}(\mathcal{H})| \leq \left(1 - \frac{4r}{(4r-1)n}\right)\binom{n}{r} \tag{1}$$

holds.

Proof: Let m denote the number of missing edges (the r-element subsets which are not edges of $\mathcal{H}$). By (1) we obtain

$$m < \frac{4r}{(4r-1)n}\binom{n}{r}.$$

Observe, that if a hypergraph contains a 1-edge-hamiltonian subgraph then one must delete at least 2 edges from it to destroy all hamiltonian chains. Therefore we count the number of occurrences of the 1-edge-hamiltonian hypergraph constructed in Theorem 13. Let $\mathcal{G}$ denote this r-uniform hypergraph on n vertices.

It is a simple matter to prove that there are $\frac{n!}{|\operatorname{Aut}(\mathcal{G})|}$ different $\mathcal{G}$ sub-hypergraphs in $\mathcal{K}_n^{(r)}$, where $\operatorname{Aut}(\mathcal{G})$ denotes the automorphism group of $\mathcal{G}$. Since every edge of $\mathcal{K}_n^{(r)}$ is contained by the same number of $\mathcal{G}$ sub-hypergraphs, the number of $\mathcal{G}$ sub-hypergraphs which contains a specified edge is

$$\frac{|\mathcal{E}(\mathcal{G})|}{\binom{n}{r}} \cdot \frac{n!}{|\operatorname{Aut}(\mathcal{G})|}.$$

Thus the number of hamiltonian chains in $\mathcal{H}$ is

$$\geq 2 \cdot \frac{n!}{|\operatorname{Aut}(\mathcal{G})|} - m\frac{|\mathcal{E}(\mathcal{G})|}{\binom{n}{r}} \cdot \frac{n!}{|\operatorname{Aut}(\mathcal{G})|} > 0,$$

our claim is proved, because by Theorem 13 $|\mathcal{E}(\mathcal{G})| = \frac{4r-1}{2r}n$. $\square$

The best known construction is due to Tuza.

Theorem 17 ([15]) *For any $n > r$ there exists an r-uniform hypergraph on n vertices with*

$$\binom{n-1}{r} + \binom{n-1}{r-2} + o(n)$$

edges.

Proof: [15] To construct this hypergraph $\mathcal{H}$, similarly to the graph case, first take a complete r-uniform hypergraph on $n-1$ vertices. Then we can add more edges containing the nth vertex v_n without creating a hamiltonian chain. If v_n is deleted from these edges, an $(r-1)$-regular hypergraph is created on $\{v_1, \dots, v_{n-1}\}$, denote this by $\mathcal{H}'$. Note that if $\mathcal{H}$ contain a hamiltonian chain, then $\mathcal{H}'$

contains an open chain of length $2r-3$. So if we construct $\mathcal{H}'$ such that it does not contain an open chain of length $2r-3$, then $\mathcal{H}$ will not contain a hamiltonian chain.

The construction of $\mathcal{H}'$ is as follows: Choose a Steiner system in which every $(r-2)$-element subset of $\{v_1, \ldots, v_{n-1}\}$ is contained precisely one $(2r-3)$-element subset. Let $S_1, \ldots, S_t$ denote these $(2r-3)$-element subsets. (Such systems exists only for certain n values, but if n is not such a value then there exists a system which almost have the above property. Such a system is good enough for us.) Take all $(r-1)$-element subsets of all S_i to be the edge set of $\mathcal{H}'$.

It is known that $t = \binom{n-1}{r-2}/\binom{2r-3}{r-2}$. (If only a nearly good system exists, then a $o(n)$ factor is added to this.) Therefore

$$|\mathcal{E}(\mathcal{H})| = \binom{n-1}{r} + \frac{\binom{n-1}{r-2}}{\binom{2r-3}{r-2}}\binom{2r-3}{r-1} = \binom{n-1}{r} + \binom{n-1}{r-2}.$$

As it was mentioned before it is enough to prove that $\mathcal{H}'$ does not contain an open chain of length $2r-3$. Suppose indirectly that it contains one. Such a chain contains $2r-2$ distinct vertices, so it is not possible that all the edges is contained by one S_i set. So the chain must contain edges contained say by S_i and S_j. Since these edges form a chain, this implies that there must be two edges $E_1 \subset S_i$ and $E_2 \subset S_j$ such that $|E_1 \cap E_2| = r-1$. This contradicts the choice of the construction of $\mathcal{H}'$. □

5 Hamiltonian chain saturated hypergraphs

Definition 18 *We say that hypergraph $\mathcal{H}$ is* hamiltonian chain saturated *if $\mathcal{H}$ does not contain a hamiltonian chain but by adding any new edge we create a hamiltonian chain in $\mathcal{H}$. Similarly, we say that $\mathcal{H}$ is* hamiltonian path saturated *if $\mathcal{H}$ does not contain an open hamiltonian chain, but by adding any new edge we create an open hamiltonian chain in $\mathcal{H}$.*

The question investigated in this section is the following: *What is the minimum number of edges in a hamiltonian path (chain) saturated hypergraph on n vertices?*

We will only deal now with the hamiltonian path saturated case. For graphs, usually the questions for hamiltonian cycle and path have similar solutions. If we have an answer for one of the questions, the other one follows easily. So although we deal now with the path case only, hopefully similar results can be proved for the chain case. However, it is important to note, that if $r > 2$ than an open hamiltonian chain and a hamiltonian chain is "not so close" then in the graph case. One can create a hamiltonian cycle from a hamiltonian path by adding a single edge. On the other hand, in the general case $r-1$ edges are needed to achieve the same goal.

The above question for graphs was raised by Bollobás [1] and solved completely for hamiltonian cycles in a series of papers. First Bondy proved in [2] that a hamiltonian cycle saturated graph on n vertices has at least $\lceil \frac{3n}{2} \rceil$ edges if $n \geq 7$. Combined results of Clark, Entrigner and Shapiro [3, 4] and Xiaohui, Wenzhou, Chengxue and Yuansheng [10] show that this bound is sharp apart from a few small values of n.

The corresponding results for the hamiltonian path saturated case can be found in [7]. These results are nearly sharp. The construction are based on Isaak's snarks and generalized Petersen graphs, which looks to be impossible to generalize for hypergraphs.

In [6] the authors prove a lower bound on the number of edges for the 3-uniform hamiltonian path saturated case.

Theorem 19 ([6]) *If $\mathcal{H}$ an r-uniform hypergraph is hamiltonian chain saturated, then $|\mathcal{E}(\mathcal{H})| \geq \binom{n}{r}/(r(n-r)+1)$.*

Proof: We prove that every r-tuple $E_0 = \{v_1, \ldots, v_r\}$ contains an $(r-1)$-element subset, which is contained by an edge of $\mathcal{H}$.

If $E_0 \in \mathcal{E}(\mathcal{H})$ then any $(r-1)$-element subset is contained by E_0 which is an edge, so the claim holds.

Now suppose that $E_0 \notin \mathcal{E}(\mathcal{H})$. Since $\mathcal{H}$ is Hamiltonian chain saturated, it does not contain a Hamiltonian chain, but adding E_0 creates one. Therefore E_0 must be an edge of this Hamiltonian chain, so it has a neighboring edge in the chain (even if it is at the end of the chain). This edge satisfies the conditions of the claim.

Using the claim we obtain that for all possible r-tuples we can find an edge that intersects the r-tuple in at least $(r-1)$ elements. However, in this way every such edge is counted $r(n-r)+1$ times. $\square$

The best construction so far has more edges then this. Even the order of magnitude is larger.

Definition 20 *Let p be a positive integer and $U_0, U_1, \ldots, U_{p+2}$ be pairwise disjoint sets of vertices such that $|U_0| = p \geq 1$ and $|U_i| \geq 3$ for $i = 1, 2, \ldots, p+2$. Define the vertex set of the hypergraph $\mathcal{H} = \mathcal{H}(U_0, U_1, \ldots, U_{p+2})$ to be $V(\mathcal{H}) = \bigcup_{i=0}^{p+2} U_i$. The edge set is defined such that the induced subhypergraph $\mathcal{H}(U_0 \cup U_i)$ is complete hypergraph for all $i = 1, 2, \ldots, p+2$. The family of all hypergraphs obtained by this construction is denoted by $\mathcal{I}_1$.*

For simplicity we will suppose that the indices of $U_1, \ldots, U_{p+2}$ are understood cyclicly so $U_{p+2+r} = U_r$, if $r > 0$.

Definition 21 *Let $\mathcal{H} \in \mathcal{I}_1$. An edge $E_0 = \{x, y, z\}$ where $x \in U_i, y, z \in U_{i+1}$ or $x \in U_i, y \in U_0, z \in U_{i+1}$ is called a jumping edge from U_i to U_{i+1}. The set of all jumping edges from U_i to U_{i+1} is denote by $J_{U_i U_{i+1}}$.*

Definition 22 *Let $\mathcal{H}(U_0, U_1, \ldots, U_{p+2}) \in \mathcal{I}_1$. Define the hypergraph $\mathcal{G} = \mathcal{G}(U_0, U_1, \ldots, U_{p+2})$ as a hypergraph with vertex set $V(\mathcal{G}) = V(\mathcal{H})$ and*

$$E(\mathcal{G}) = E(\mathcal{H}) \cup J_{U_1 U_2} \cup \cdots \cup J_{U_{p+1} U_{p+2}} \cup J_{U_{p+2} U_1}.$$

The family of hypergraphs obtained by this construction is denoted by $\mathcal{J}_1$.

Theorem 23 ([6]) *Let $\mathcal{G} \in \mathcal{J}_1$ and $p = 1$. If $|U_1| < 2|U_3| - 1, |U_2| < 2|U_1| - 1$ and $|U_3| < 2|U_2| - 1$ holds then $\mathcal{G}$ is hamiltonian chain saturated.*

Corollary 24 *For any $n \geq 7$ there exists a 3-uniform hypergraph with $\frac{2}{27}n^3 + o(n^3)$ edges.*

The authors conjecture that the right order of magnitude is n^{r-1} in the general case. This question and the corresponding question for the hamiltonian chain saturated case is still open.

References

[1] B. Bollobás. *Extremal graph theory*, volume 11 of *London Mathematical Society Monographs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978.

[2] J. A. Bondy. Variations on the Hamiltonian theme. *Canad. Math. Bull.*, 15:57–62, 1972.

[3] L. Clark and R. Entringer. Smallest maximally non-Hamiltonian graphs. *Period. Math. Hungar.*, 14(1):57–68, 1983.

[4] L. H. Clark, R. C. Entringer, and H. D. Shapiro. Smallest maximally non-Hamiltonian graphs. II. *Graphs Combin.*, 8(3):225–231, 1992.

[5] G. A. Dirac. Some theorems on abstract graphs. *Proc. London Math. Soc. (3)*, 2:69–81, 1952.

[6] A. Dudek and G. Y. Katona. The generalization of zelinka constructions for 3-uniform hypergraphs. manuscript.

[7] A. Dudek, G. Y. Katona, and A. P. Wojda. Hamiltonian path saturated graphs with small size. to appear in Discrete Appllied Math.

[8] P. Frankl and G. Y. Katona. k-edge-hamiltonian hypergraphs. submitted to J. of Graphs Theory in 2002.

[9] G. Y. Katona and H. A. Kierstead. Hamiltonian chains in hypergraphs. *J. Graph Theory*, 30(3):205–212, 1999.

[10] X. Lin, W. Jiang, C. Zhang, and Y. Yang. On smallest maximally non-Hamiltonian graphs. *Ars Combin.*, 45:263–270, 1997.

[11] H. Müller. privat communication. 1998.

[12] Ö. Øre. Arc coverings of graphs. *Ann. Mat. Pura Appl. (4)*, 55:315–321, 1961.

[13] M. Paoli, W. W. Wong, and C. K. Wong. Minimum k-Hamiltonian graphs. II. *J. Graph Theory*, 10(1):79–95, 1986.

[14] A. Ruciński, V. Rödl, and E. Szemerédi. privat communication. 2001.

[15] Z. Tuza. private communication.

[16] W. W. Wong and C. K. Wong. Minimum k-Hamiltonian graphs. *J. Graph Theory*, 8(1):155–165, 1984.