Minicourse on parameterized algorithms and complexity

Part 4: Treewidth

Dániel Marx

November 9, 2016

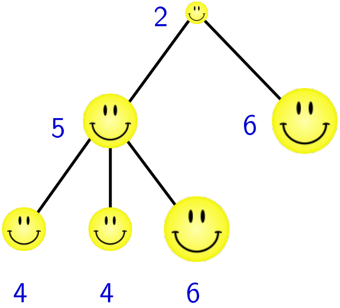
Treewidth

- Treewidth: a notion of "treelike" graphs.
- Some combinatorial properties.
- Algorithmic results.
  - Algorithms on graphs of bounded treewidth.
  - Applications for other problems.

The Party Problem

Party Problem

Problem:	Invite some colleagues for a party.
Maximize:	The total fun factor of the invited people.
Constraint:	Everyone should be having fun.



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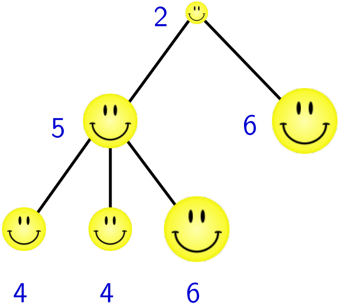

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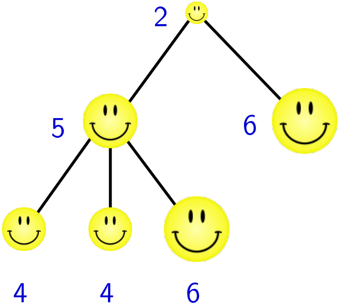

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- **Task:** Find an independent set of maximum weight.

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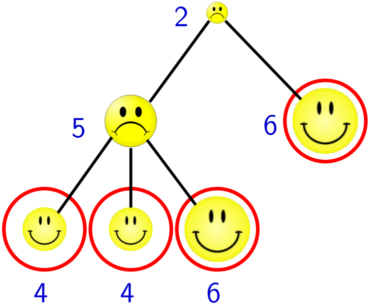

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Solving the Party Problem

Dynamic programming paradigm:
We solve a large number of subproblems that depend on each other. The answer is a single subproblem.

Subproblems:

T_v: the subtree rooted at v.
$A[v]$: max. weight of an independent set in T_v
$B[v]$: max. weight of an independent set in T_v that does not contain v

Goal: determine $A[r]$ for the root r.

Solving the Party Problem

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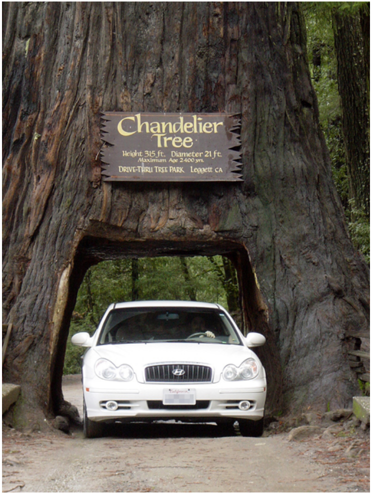
Recurrence:

Assume $v_1, \ldots, v_k$ are the children of v. Use the recurrence relations

$$B[v] = \sum_{i=1}^{k} A[v_i]$$
$$A[v] = \max\{B[v]\ ,\ w(v) + \sum_{i=1}^{k} B[v_i]\}$$

The values $A[v]$ and $B[v]$ can be calculated in a bottom-up order (the leaves are trivial).


Chandelier Tree
Height 315 ft. Diameter 21 ft.
Maximum Age 2400 yrs.
DRIVE-THRU TREE PARK Leggett CA

Treewidth

Generalizing trees

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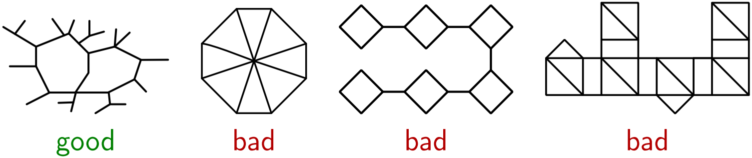
1. Number of cycles is bounded.



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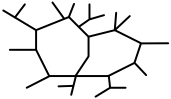

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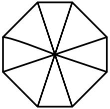

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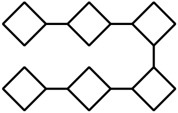

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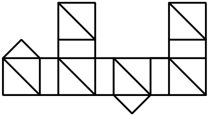

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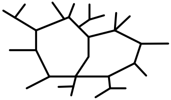
2. Removing a bounded number of vertices makes it acyclic.



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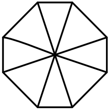

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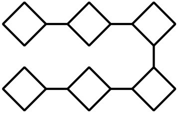

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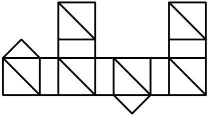

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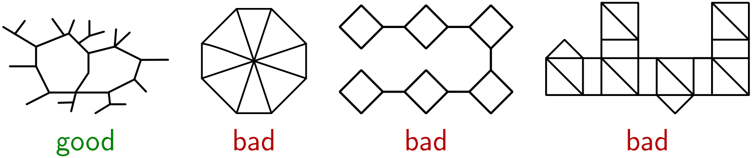

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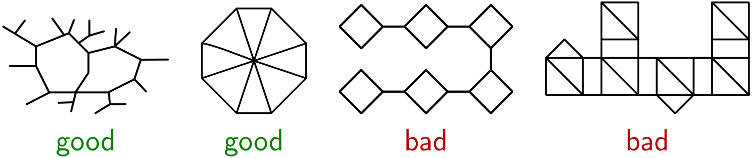
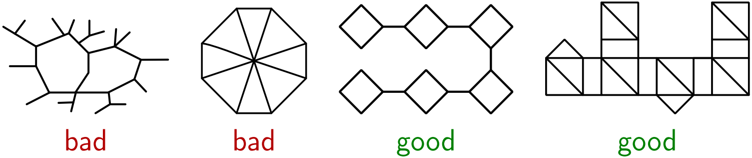


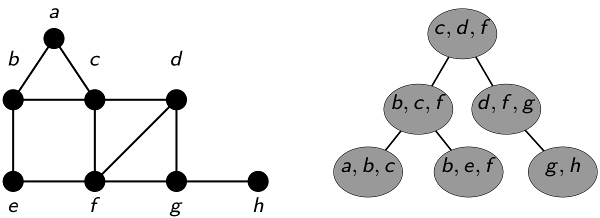
3. Bounded-size parts connected in a tree-like way.



Treewidth — a measure of "tree-likeness"

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. If u and v are neighbors, then there is a bag containing both of them.
2. For every v, the bags containing v form a connected subtree.



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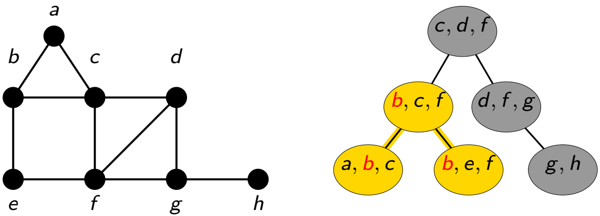

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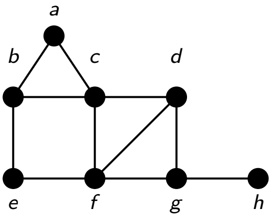

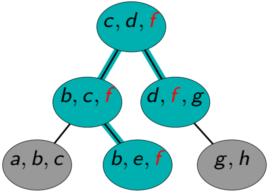

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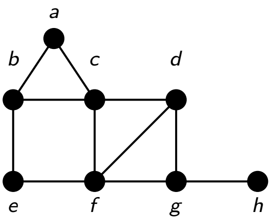
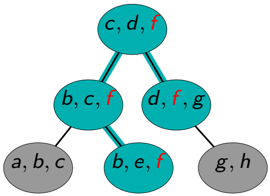
Width of the decomposition: largest bag size -1.

treewidth: width of the best decomposition.





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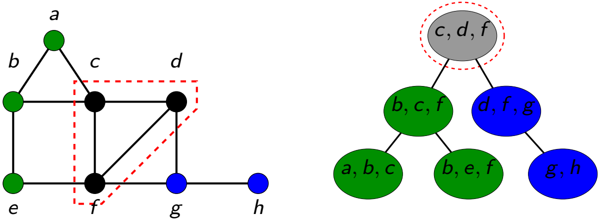

Each bag is a separator.

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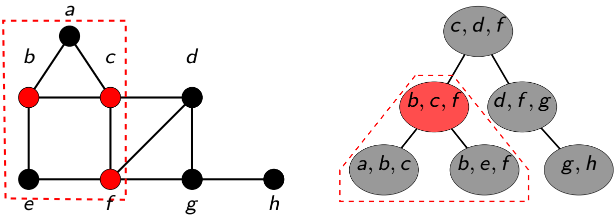
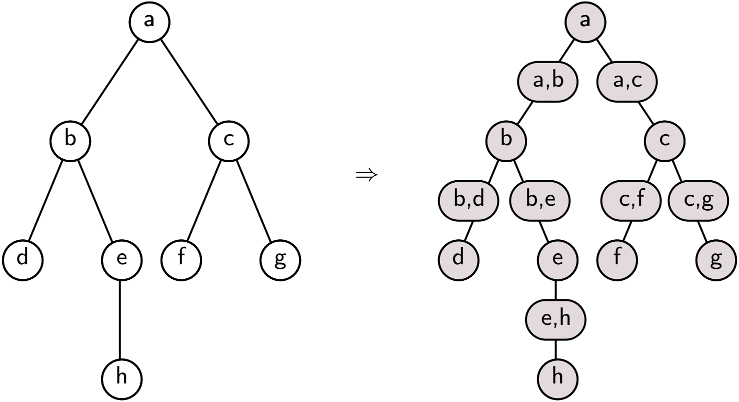


A subtree communicates with the outside world only via the root of the subtree.

Treewidth

Fact: treewidth $= 1 \iff$ graph is a forest



Exercise: A cycle cannot have a tree decomposition of width 1.

Treewidth — outline

1. Basic algorithms
2. Combinatorial properties
3. Applications

Finding tree decompositions

Hardness:

Theorem [Arnborg, Corneil, Proskurowski 1987]

It is NP-hard to determine the treewidth of a graph (given a graph G and an integer w, decide if the treewidth of G is at most w).

Fixed-parameter tractability:

Theorem [Bodlaender 1996]

There is a $2^{O(w^3)} \cdot n$ time algorithm that finds a tree decomposition of width w (if exists).

Consequence:
If we want an FPT algorithm parameterized by treewidth w of the input graph, then we can assume that a tree decomposition of width w is available.

Finding tree decompositions — approximately

Sometimes we can get better dependence on treewidth using approximation.

FPT approximation:

Theorem [Robertson and Seymour]

There is a $O(3^{3w} \cdot w \cdot n^2)$ time algorithm that finds a tree decomposition of width $4w + 1$, if the treewidth of the graph is at most w.

Polynomial-time approximation:

Theorem [Feige, Hajiaghayi, Lee 2008]

There is a polynomial-time algorithm that finds a tree decomposition of width $O(w\sqrt{\log w})$, if the treewidth of the graph is at most w.

Weighted Max Independent Set and treewidth

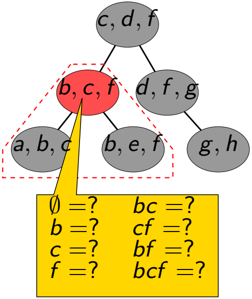
Theorem

Given a tree decomposition of width w, Weighted Max Independent Set can be solved in time $O(2^w \cdot w^{O(1)} \cdot n)$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

Generalizing our solution for trees:

Instead of computing 2 values $A[v]$, $B[v]$ for each **vertex** of the graph, we compute $2^{|B_x|} \leq 2^{w+1}$ values for each bag B_x.



$M[x, S]$:
the max. weight of an independent set $I \subseteq V_x$ with $I \cap B_x = S$.

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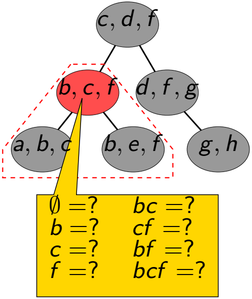

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How to determine $M[x, S]$ if all the values are known for the children of x?

Nice tree decompositions

Definition

A rooted tree decomposition is **nice** if every node x is one of the following 4 types:

- **Leaf:** no children, $|B_x| = 1$
- **Introduce:** 1 child y with $B_x = B_y \cup \{v\}$ for some vertex v
- **Forget:** 1 child y with $B_x = B_y \setminus \{v\}$ for some vertex v
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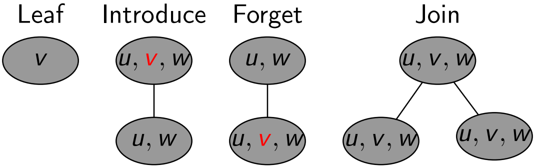

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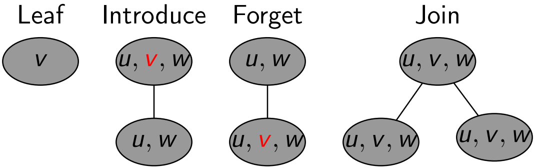
Theorem

A tree decomposition of width w and n nodes can be turned into a nice tree decomposition of width w and $O(wn)$ nodes in time $O(w^2 n)$.

Weighted Max Independent Set
and nice tree decompositions

- **Leaf:** no children, $|B_x| = 1$
  Trivial!
- **Introduce:** 1 child y with $B_x = B_y \cup \{v\}$ for some vertex v

$$m[x,S] = \begin{cases} m[y,S] & \text{if } v \notin S, \\ m[y, S \setminus \{v\}] + w(v) & \text{if } v \in S \text{ but } v \text{ has no neighbor in } S, \\ -\infty & \text{if } S \text{ contains } v \text{ and its neighbor.} \end{cases}$$



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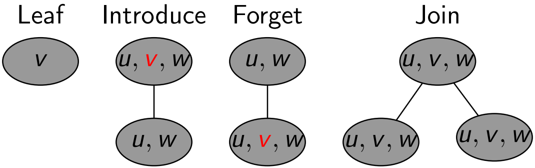

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There are at most $2^{w+1} \cdot n$ subproblems $m[x, S]$ and each subproblem can be solved in $w^{O(1)}$ time (assuming the children are already solved).

$\Downarrow$

Running time is $O(2^w \cdot w^{O(1)} \cdot n)$.

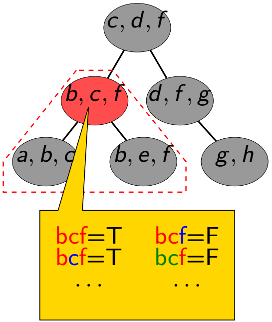
3-Coloring and tree decompositions

Theorem

Given a tree decomposition of width w, 3-Coloring can be solved in $O(3^w \cdot w^{O(1)} \cdot n)$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

For every node x and coloring $c : B_x \to \{1, 2, 3\}$, we compute the Boolean value $E[x, c]$, which is true if and only if c can be extended to a proper 3-coloring of V_x.



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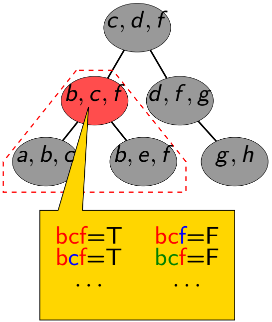

How to determine $E[x, c]$ if all the values are known for the children of x?

3-Coloring and nice tree decompositions

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- **Introduce:** 1 child y with $B_x = B_y \cup \{v\}$ for some vertex v
  If $c(v) \neq c(u)$ for every neighbor u of v, then
  $E[x, c] = E[y, c']$, where c' is c restricted to B_y.
- **Forget:** 1 child y with $B_x = B_y \setminus \{v\}$ for some vertex v
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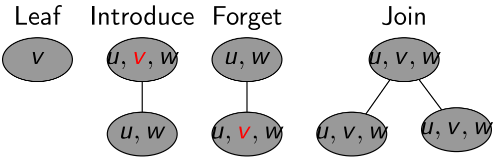

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There are at most $3^{w+1} \cdot n$ subproblems $E[x, c]$ and each subproblem can be solved in $w^{O(1)}$ time (assuming the children are already solved).

$\Rightarrow$ Running time is $O(3^w \cdot w^{O(1)} \cdot n)$.

$\Rightarrow$ 3-Coloring is FPT parameterized by treewidth.

Monadic Second Order Logic

Extended Monadic Second Order Logic (EMSO)

A logical language on graphs consisting of the following:

- Logical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $=$, $\neq$
- quantifiers $\forall$, $\exists$ over vertex/edge variables
- predicate $\mathrm{adj}(u, v)$: vertices u and v are adjacent
- predicate $\mathrm{inc}(e, v)$: edge e is incident to vertex v
- quantifiers $\forall$, $\exists$ over vertex/edge set variables
- $\in$, $\subseteq$ for vertex/edge sets

Example:
The formula

$$\exists C \subseteq V \exists v_0 \in C \forall v \in C \; \exists u_1, u_2 \in C (u_1 \neq u_2 \wedge \mathrm{adj}(u_1, v) \wedge \mathrm{adj}(u_2, v))$$

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is true on graph G if and only if G has a cycle.

Courcelle's Theorem

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If a graph property can be expressed in EMSO, then for every fixed $w \geq 1$, there is a linear-time algorithm for testing this property on graphs having treewidth at most w.

Note: The constant depending on w can be very large (double, triple exponential etc.), therefore a direct dynamic programming algorithm can be more efficient.

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If we can express a property in EMSO, then we immediately get that testing this property is FPT parameterized by the treewidth w of the input graph.

Can we express 3-Coloring and Hamiltonian Cycle in EMSO?

Using Courcelle's Theorem

3-Coloring

$\exists C_1, C_2, C_3 \subseteq V \left(\forall v \in V \left(v \in C_1 \vee v \in C_2 \vee v \in C_3\right)\right) \wedge \left(\forall u, v \in V \operatorname{adj}(u, v) \rightarrow (\neg(u \in C_1 \wedge v \in C_1) \wedge \neg(u \in C_2 \wedge v \in C_2) \wedge \neg(u \in C_3 \wedge v \in C_3))\right)$

Using Courcelle's Theorem

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$\exists C_1, C_2, C_3 \subseteq V \big(\forall v \in V\, (v \in C_1 \vee v \in C_2 \vee v \in C_3)\big) \wedge \big(\forall u, v \in V\, \mathrm{adj}(u, v) \rightarrow (\neg(u \in C_1 \wedge v \in C_1) \wedge \neg(u \in C_2 \wedge v \in C_2) \wedge \neg(u \in C_3 \wedge v \in C_3))\big)$

Hamiltonian Cycle

$\exists H \subseteq E\big(\mathrm{spanning}(H) \wedge (\forall v \in V\, \mathrm{degree2}(H, v))\big)$

$\mathrm{degree0}(H, v) := \neg\exists e \in H\, \mathrm{inc}(e, v)$

$\mathrm{degree1}(H, v) := \neg\mathrm{degree0}(H, v) \wedge \big(\neg\exists e_1, e_2 \in H\, (e_1 \neq e_2 \wedge \mathrm{inc}(e_1, v) \wedge \mathrm{inc}(e_2, v))\big)$

$\mathrm{degree2}(H, v) := \neg\mathrm{degree0}(H, v) \wedge \neg\mathrm{degree1}(H, v) \wedge \big(\neg\exists e_1, e_2, e_3 \in H\, (e_1 \neq e_2 \wedge e_2 \neq e_3 \wedge e_1 \neq e_3 \wedge \mathrm{inc}(e_1, v) \wedge \mathrm{inc}(e_2, v) \wedge \mathrm{inc}(e_3, v))\big)$

$\mathrm{spanning}(H) := \forall u, v \in V\, \exists P \subseteq H\, \forall x \in V\, \big(((x = u \vee x = v) \wedge \mathrm{degree1}(P, x)) \vee (x \neq u \wedge x \neq v \wedge (\mathrm{degree0}(P, x) \vee \mathrm{degree2}(P, x)))\big)$

Using Courcelle's Theorem

Two ways of using Courcelle's Theorem:

1. The problem can be described by a single formula (e.g, 3-Coloring, Hamiltonian Cycle).

   $\Rightarrow$ Problem can be solved in time $f(w) \cdot n$ for graphs of treewidth at most w, i.e., FPT parameterized by treewidth.

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2. The problem can be described by a formula for each value of the parameter k.

   Example: For each k, having a cycle of length exactly k can be expressed as

   $$\exists v_1, \ldots, v_k \in V\,((v_1 \neq v_2) \wedge (v_1 \neq v_3) \wedge \ldots (v_{k-1} \neq v_k))$$
   $$\wedge \mathrm{adj}(v_{k-1}, v_k) \wedge \mathrm{adj}(v_k, v_1)).$$

   $\Rightarrow$ Problem can be solved in time $f(k, w) \cdot n$ for graphs of treewidth w, i.e., FPT parameterized with combined parameter k and treewidth w.

Subgraph Isomorphism

Subgraph Isomorphism

Input: graphs H and G
Find: a subgraph of G isomorphic to H.

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For each H, we can construct a formula ϕ_H that expresses "G has a subgraph isomorphic to H" (similarly to the k-cycle on the previous slide).

$\Rightarrow$ By Courcelle's Theorem, Subgraph Isomorphism can be solved in time $f(H, w) \cdot n$ if G has treewidth at most w.

Subgraph Isomorphism

Subgraph Isomorphism

Input: graphs H and G

Find: a subgraph of G isomorphic to H.

Since there is only a finite number of simple graphs on k vertices, Subgraph Isomorphism can be solved in time $f(k, w) \cdot n$ if H has k vertices and G has treewidth at most w.

Theorem

Subgraph Isomorphism is FPT parameterized by combined parameter $k := |V(H)|$ and the treewidth w of G.

MSO on words

Theorem [Büchi, Elgot, Trakhtenbrot 1960]

If a language $L \subseteq \Sigma^*$ can be defined by an MSO formula ϕ using the relation $<$, then L is regular.

Example: $a^* bc^*$ is defined by

$$\exists x : P_b(x) \land (\forall y : (y < x) \to P_a(y)) \land (\forall y : (x < y) \to P_c(y)).$$

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We prove a more general statement for formulas $\phi(w, X_1, \dots, X_k)$ and words over $\Sigma \cup \{0,1\}^k$, where X_i is a subset of symbols of w.

Induction over the structure of ϕ:

- FSM for $\neg\phi(w)$, given FSM for $\phi(w)$.
- FSM for $\phi_1(w) \wedge \phi_2(w)$, given FSMs for $\phi_1(w)$ and $\phi_2(w)$.
- FSM for $\exists X \phi(w, X)$, given FSM for $\phi(w, X)$.
- etc.

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Proving Courcelle's Theorem:

- Generalize from words to trees.
- A width-k tree decomposition can be interpreted as a tree over an alphabet of size $f(k)$.
- Formula $\Rightarrow$ tree automata.

Algorithms — overview

- Algorithms exploit the fact that a subtree communicates with the rest of the graph via a single bag.
- Key point: defining the subproblems.
- Courcelle's Theorem makes this process automatic for many problems.
- There are notable problems that are easy for trees, but hard for bounded-treewidth graphs.

Treewidth — outline

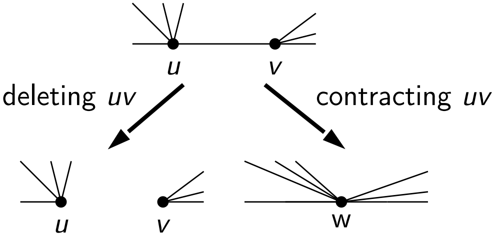
1. Basic algorithms
2. Combinatorial properties
3. Applications

Minor

An operation similar to taking subgraphs:

Definition

Graph H is a **minor** of G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.



Properties of treewidth

Fact: Treewidth does not increase if we delete edges, delete vertices, or contract edges.

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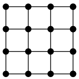
Fact: The treewidth of the k-clique is $k-1$.

Fact: For every $k \geq 2$, the treewidth of the $k \times k$ grid is exactly k.



The Cops and Robber game

Game: k cops try to capture a robber in the graph.

- In each step, the cops can move from vertex to vertex arbitrarily with helicopters.
- The robber moves infinitely fast on the edges, and sees where the cops will land.

Theorem [Seymour and Thomas 1993]

$k+1$ cops can win the game $\iff$ the treewidth of the graph is at most k.

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Consequence 1: Algorithms

The winner of the game can be determined in time $n^{O(k)}$ using standard techniques (there are at most n^k positions for the cops)

$$\Downarrow$$

For every fixed k, it can be checked in polynomial-time if treewidth is at most k.

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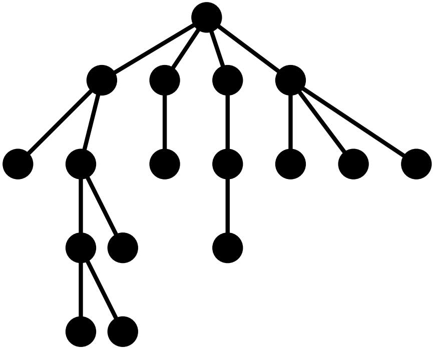
Consequence 2: Lower bounds

Exercise 1:
Show that the treewidth of the $k \times k$ grid is at least $k-1$.
(E.g., robber can win against $k-1$ cops.)

Exercise 2:
Show that the treewidth of the $k \times k$ grid is at least k.
(E.g., robber can win against k cops.)

The Cops and Robber game

Example: 2 cops have a winning strategy in a tree.



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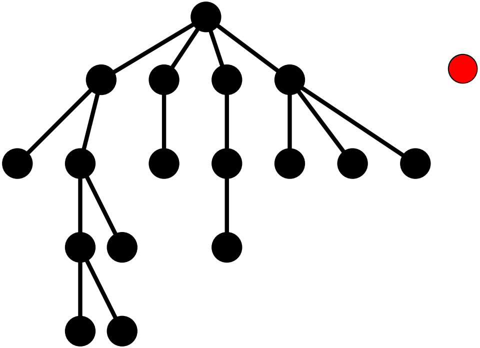

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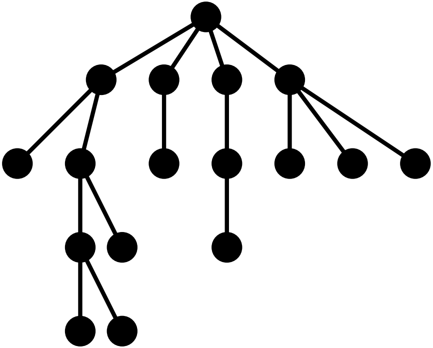

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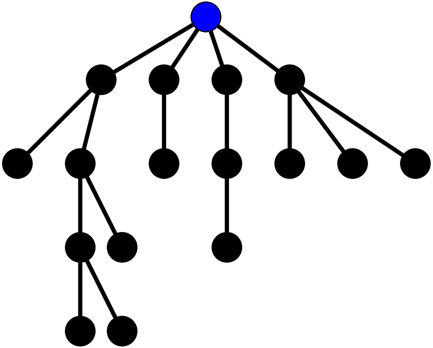

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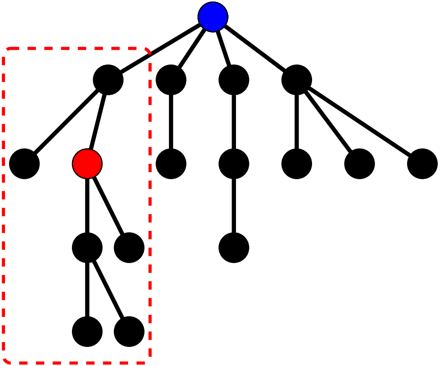

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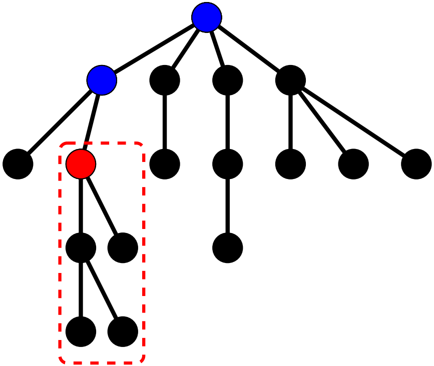

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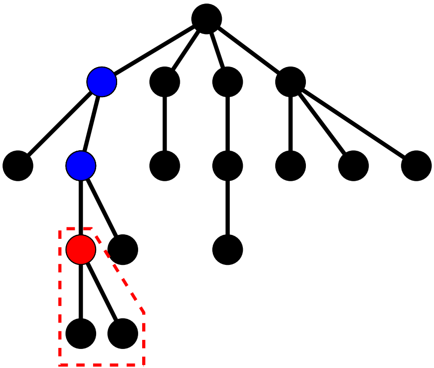

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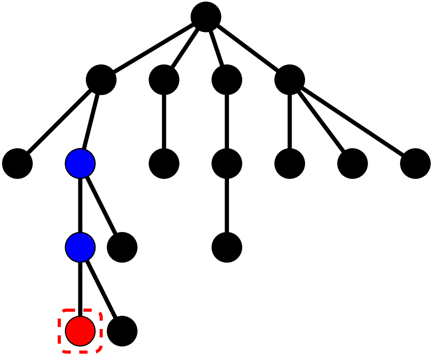

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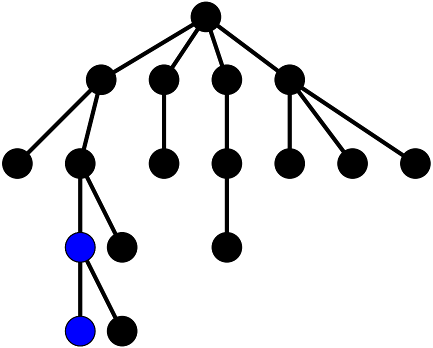
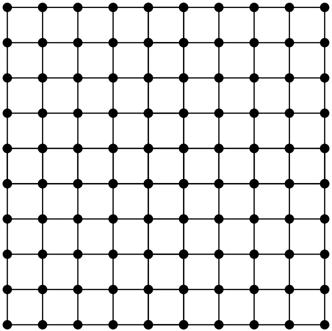


Excluded Grid Theorem

Excluded Grid Theorem [Diestel et al. 1999]

If the treewidth of G is at least $k^{4k^2(k+2)}$, then G has a $k \times k$ grid minor.



(A $k^{O(1)}$ bound was achieved recently [Chekuri and Chuznoy 2014]!)

Excluded Grid Theorem

Excluded Grid Theorem [Diestel et al. 1999]

If the treewidth of G is at least $k^{4k^2(k+2)}$, then G has a $k \times k$ grid minor.

Observation: Every planar graph is the minor of a sufficiently large grid.

Consequence

If H is planar, then every H-minor free graph has treewidth at most $f(H)$.

Excluded Grid Theorem

Excluded Grid Theorem [Diestel et al. 1999]

If the treewidth of G is at least $k^{4k^2(k+2)}$, then G has a $k \times k$ grid minor.

A large grid minor is a "witness" that treewidth is large, but the relation is approximate:

No $k \times k$ grid minor $\Longrightarrow$ tree decomposition of width $< f(k)$

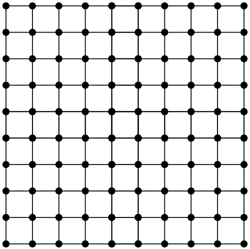
tree decomposition of width $< f(k)$ $\Longrightarrow$ no $f(k) \times f(k)$ grid minor

Planar Excluded Grid Theorem

For planar graphs, we get linear instead of exponential dependence:

Theorem [Robertson, Seymour, Thomas 1994]

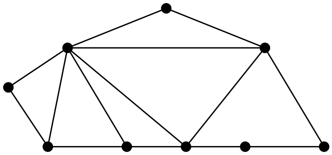
Every **planar graph** with treewidth at least $5k$ has a $k \times k$ grid minor.



Outerplanar graphs

Definition

A planar graph is **outerplanar** if it has a planar embedding where every vertex is on the infinite face.



Fact

Every outerplanar graph has treewidth at most 2.

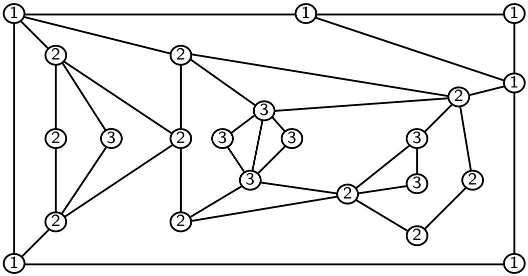
$\Rightarrow$ Every outerplanar graph is subgraph of a series-parallel graph.

k-outerplanar graphs

Given a planar embedding, we can define **layers** by iteratively removing the vertices on the infinite face.

Definition

A planar graph is k-**outerplanar** if it has a planar embedding having at most k layers.



Fact

Every k-outerplanar graph has treewidth at most $3k + 1$.

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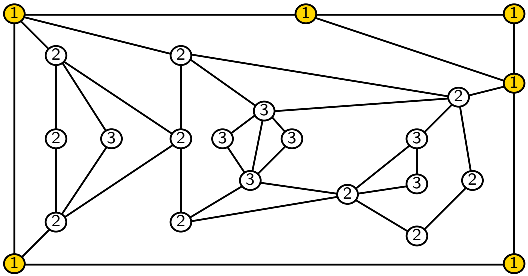

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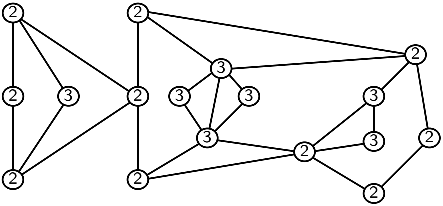

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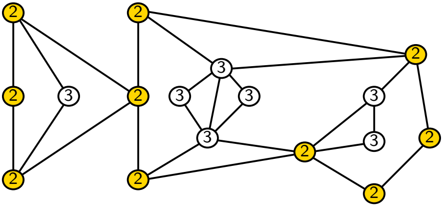

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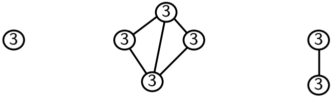

Fact

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Treewidth — outline

1. Basic algorithms
2. Combinatorial properties
3. Applications
   - The shifting technique
   - Bidimensionality

Approximation schemes

Definition

A **polynomial-time approximation scheme (PTAS)** for a problem P is an algorithm that takes an instance of P and a rational number $\epsilon > 0$,

- always finds a $(1 + \epsilon)$-approximate solution,
- the running time is polynomial in n for every fixed $\epsilon > 0$.

Typical running times: $2^{1/\epsilon} \cdot n$, $n^{1/\epsilon}$, $(n/\epsilon)^2$, n^{1/ϵ^2}.

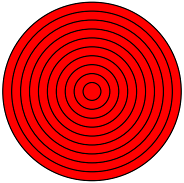
Some classical problems that have a PTAS:

- Independent Set for planar graphs
- TSP in the Euclidean plane
- Steiner Tree in planar graphs
- Knapsack

Baker's shifting strategy for PTAS

Theorem

There is a $2^{O(1/\epsilon)} \cdot n$ time PTAS for Independent Set for planar graphs.

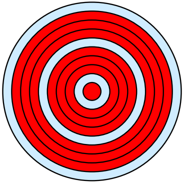


- Let $D := 1/\epsilon$. For a fixed $0 \leq s < D$, delete every layer L_i with $i = s \pmod{D}$

Baker's shifting strategy for PTAS

Theorem

There is a $2^{O(1/\epsilon)} \cdot n$ time PTAS for INDEPENDENT SET for planar graphs.



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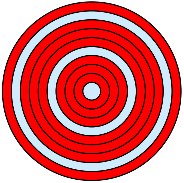

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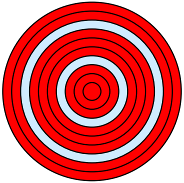

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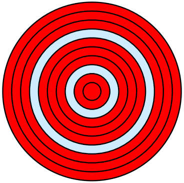

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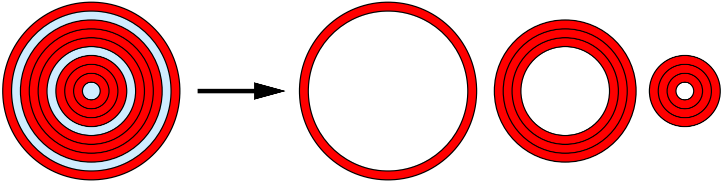

- Let $D := 1/\epsilon$. For a fixed $0 \leq s < D$, delete every layer L_i with $i = s \pmod{D}$
- The resulting graph is D-outerplanar, hence it has treewidth at most $3D + 1 = O(1/\epsilon)$.
- Using the $2^{O(\text{tw})} \cdot n$ time algorithm for INDEPENDENT SET, the problem on the D-outerplanar graph can be solved in time $2^{O(1/\epsilon)} \cdot n$.

Baker's shifting strategy for PTAS

Theorem

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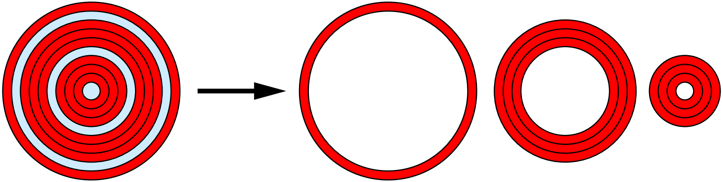


We do this for every $0 \leq s < D$:
for at least one value of s, we delete
at most $1/D = \epsilon$ fraction of the solution

$\Downarrow$

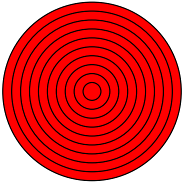
We get a $(1+\epsilon)$-approximate solution.

Baker's shifting strategy for FPT

Subgraph Isomorphism

Input: graphs H and G

Find: a subgraph G isomorphic to H.



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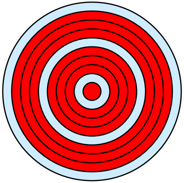

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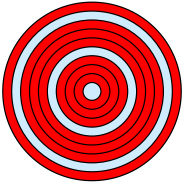

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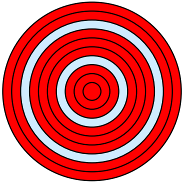

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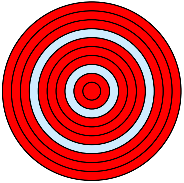

- For a fixed $0 \leq s < k+1$, delete every layer L_i with $i = s \pmod{k+1}$

Baker's shifting strategy for FPT

Subgraph Isomorphism

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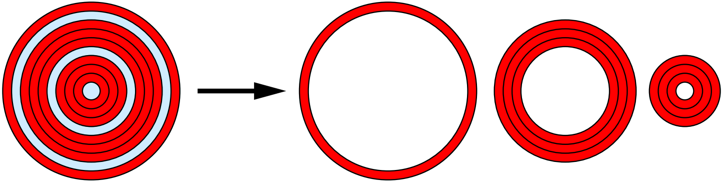

- For a fixed $0 \leq s < k+1$, delete every layer L_i with $i = s \pmod{k+1}$
- The resulting graph is k-outerplanar, hence it has treewidth at most $3k+1$.
- Using the $f(k, \text{tw}) \cdot n$ time algorithm for Subgraph Isomorphism, the problem can be solved in time $f(k, 3k+1) \cdot n$.

Baker's shifting strategy for FPT

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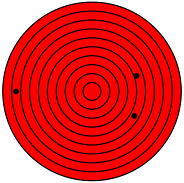

We do this for every $0 \leq s < k+1$:
for at least one value of s, we do not delete
any of the k vertices of the solution

$\Downarrow$

We find a copy of H in G if there is one.

Baker's shifting strategy for FPT

SUBGRAPH ISOMORPHISM

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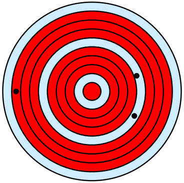

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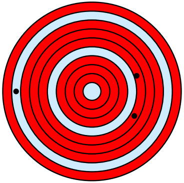

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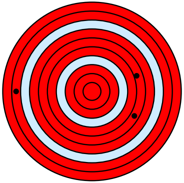

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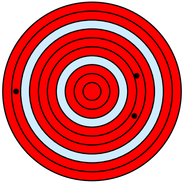


Theorem

Subgraph Isomorphism for planar graphs is FPT parameterized by $k := |V(H)|$.

Baker's shifting strategy for FPT

- The technique is very general, works for many problems on planar graphs:
  - Independent Set
  - Vertex Cover
  - Dominating Set
  - ...
- More generally: First-Order Logic problems.
- But for some of these problems, much better techniques are known (see the following slides).

Bidimensionality

A powerful framework for efficient algorithms on planar graphs.

Setup:

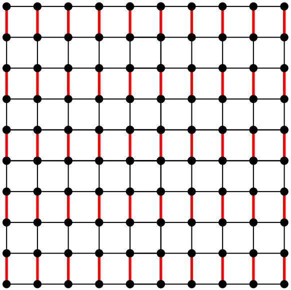
- Let $x(G)$ be some graph invariant (i.e., an integer associated with each graph).
- Given G and k, we want to decide if $x(G) \leq k$ (or $x(G) \geq k$).
- Typical examples:
  - Maximum independent set size.
  - Minimum vertex cover size.
  - Length of the longest path.
  - Minimum dominating set size.
  - Minimum feedback vertex set size.

Bidimensionality [Demaine, Fomin, Hajiaghayi, Thilikos 2005]

For many natural invariants, we can do this in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$ on planar graphs.

Bidimensionality for Vertex Cover

Observation: If the treewidth of a planar graph G is at least $5\sqrt{2k}$
$\Rightarrow$ It has a $\sqrt{2k} \times \sqrt{2k}$ grid minor (Planar Excluded Grid Theorem)
$\Rightarrow$ The grid has a matching of size k
$\Rightarrow$ Vertex cover size is at least k in the grid.
$\Rightarrow$ Vertex cover size is at least k in G.



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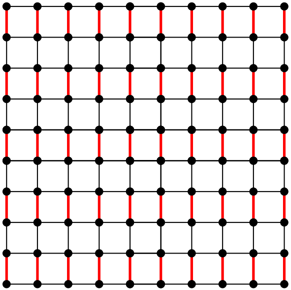
$\Rightarrow$ The grid has a matching of size k
$\Rightarrow$ Vertex cover size is at least k in the grid.
$\Rightarrow$ Vertex cover size is at least k in G.

We use this observation to solve VERTEX COVER on planar graphs:

- Set $w := 5\sqrt{2k}$.
- Find a 4-approximate tree decomposition.
  - If treewidth is at least w: we answer "vertex cover is $\geq k$."
  - If we get a tree decomposition of width $4w$, then we can solve the problem in time $2^{O(w)} \cdot n^{O(1)} = 2^{O(\sqrt{k})} \cdot n^{O(1)}$.

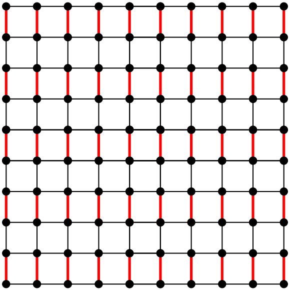


Bidimensionality

Definition

A graph invariant $x(G)$ is **minor-bidimensional** if

- $x(G') \leq x(G)$ for every minor G' of G, and
- If G_k is the $k \times k$ grid, then $x(G_k) \geq ck^2$ (for some constant $c > 0$).



Examples: minimum vertex cover, length of the longest path, feedback vertex set are minor-bidimensional.

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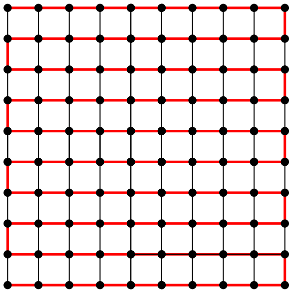

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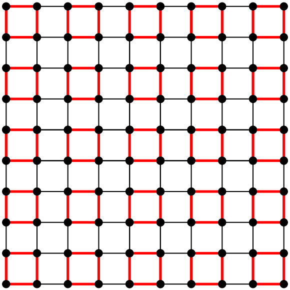

Examples: minimum vertex cover, length of the longest path, feedback vertex set are minor-bidimensional.

Bidimensionality (cont.)

We can answer "$x(G) \geq k$?" for a minor-bidimensional invariant the following way:

- Set $w := c\sqrt{k}$ for an appropriate constant c.
- Use the 4-approximation tree decomposition algorithm.
  - If treewidth is at least w: $x(G)$ is at least k.
  - If we get a tree decomposition of width $4w$, then we can solve the problem using dynamic programming on the tree decomposition.

Running time:

- If we can solve the problem on tree decomposition of width w in time $2^{O(w)} \cdot n^{O(1)}$, then the running time is $2^{O(\sqrt{k})} \cdot n^{O(1)}$.
- If we can solve the problem on tree decomposition of width w in time $w^{O(w)} \cdot n^{O(1)}$, then the running time is $2^{O(\sqrt{k} \log k)} \cdot n^{O(1)}$.

Contraction bidimensionality

Definition

A graph invariant $x(G)$ is **minor-bidimensional** if

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Exercise: DOMINATING SET is **not** minor-bidimensional.

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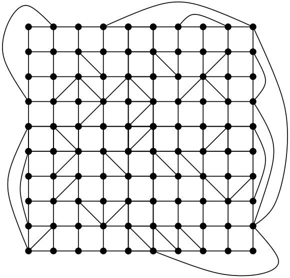
Exercise: DOMINATING SET is **not** minor-bidimensional.

We fix the problem by allowing only contractions but not edge/vertex deletions.

Contraction bidimensionality

Theorem

Every **planar graph** with treewidth at least $5k$ can be contracted to a **partially triangulated** $k \times k$ grid.

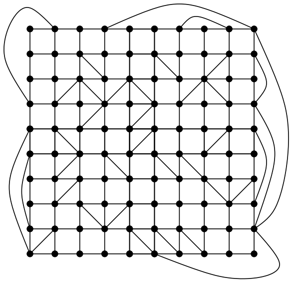


Contraction bidimensionality

Definition

A graph invariant $x(G)$ is **contraction-bidimensional** if

- $x(G') \leq x(G)$ for every contraction G' of G, and
- If G_k is a $k \times k$ partially triangulated grid, then $x(G_k) \geq ck^2$ (for some constant $c > 0$).



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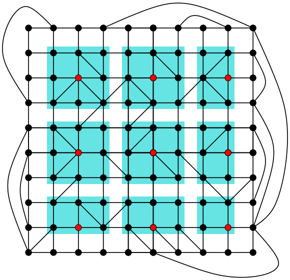

Example: minimum dominating set, maximum independent set are contraction-bidimensional.

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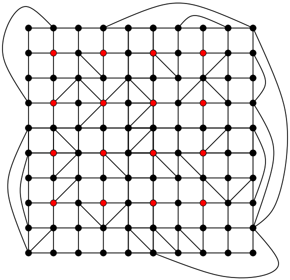

Example: minimum dominating set, maximum independent set are contraction-bidimensional.

Bidimensionality for DOMINATING SET

The size of a minimum dominating set is a **contraction bidimensional** invariant: we need at least $(k-2)^2/9$ vertices to dominate all the internal vertices of a partially triangulated $k \times k$ grid (since a vertex can dominate at most 9 internal vertices).

Theorem

Given a tree decomposition of width w, DOMINATING SET can be solved in time $3^w \cdot w^{O(1)} \cdot n^{O(1)}$.

Solving DOMINATING SET on planar graphs:

- Set $w := 5(3\sqrt{k}+2)$.
- Use the 4-approximation tree decomposition algorithm.
  - If treewidth is at least w: we answer 'dominating set is $\geq k$'.
  - If we get a tree decomposition of width $4w$, then we can solve the problem in time $3^w \cdot n^{O(1)} = 2^{O(\sqrt{k})} \cdot n^{O(1)}$.

Treewidth

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. If u and v are neighbors, then there is a bag containing both of them.
2. For every v, the bags containing v form a connected subtree.

Width of the decomposition: largest bag size -1.

treewidth: width of the best decomposition.