Minicourse on parameterized algorithms and complexity

Part 2: Iterative compression

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(slides by Marek Cygan)

November 3, 2016

What iterative compression is?

Iterative compression — main idea

Recursive approach exploiting instance structure exposed by a bit oversized solution.

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Iterative compression — main idea

Recursive approach exploiting instance structure exposed by a bit oversized solution.

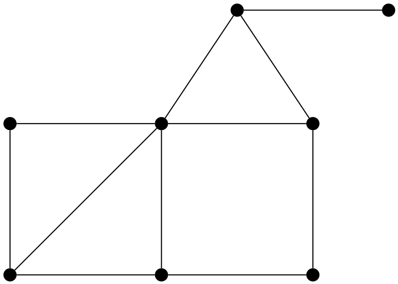
Solution compression:

1. First, apply some simple trick so that you can assume that a slightly too large solution is available.
2. Then exploit the structure it imposes on the input graph to construct an optimal solution.

Vertex Cover

Vertex Cover

Input: undirected G, integer k
Question: is there a subset $X \subseteq V(G)$ of size at most k such that for each $uv \in E(G)$ we have $\{u, v\} \cap X \neq \emptyset$.



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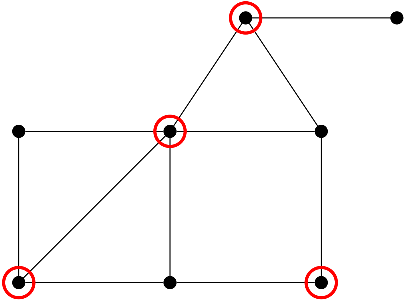

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Vertex Cover Compression

Input: undirected G, integer k,
vertex cover $Z \subseteq V(G)$ of size at most $2k$

Question: is there a vertex cover of size at most k?

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Vertex Cover Compression

Input: undirected G, integer k,
vertex cover $Z \subseteq V(G)$ of size at most $2k$

Question: is there a vertex cover of size at most k?

- Where do we get Z from?
- How do we use Z?

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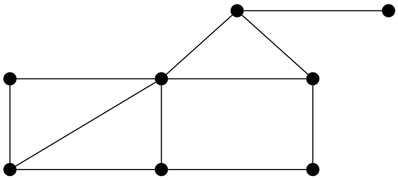
Use polynomial time 2-approximation:



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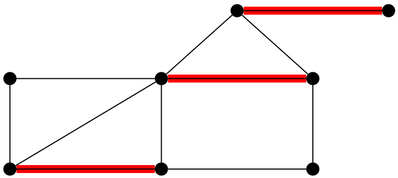
Use polynomial time 2-approximation:

- Find any inclusionwise maximal matching M.



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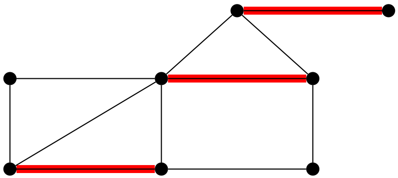
- If $|M| > k$, then no VC of size $\leqslant k$ exists.



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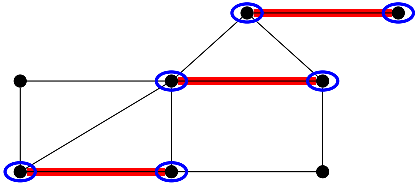
Where do we get Z from?

Use polynomial time 2-approximation:

- Find any inclusionwise maximal matching M.
- If $|M| > k$, then no VC of size $\leqslant k$ exists.
- Otherwise, set $Z = V(M)$, we have $|Z| \leqslant 2k$.



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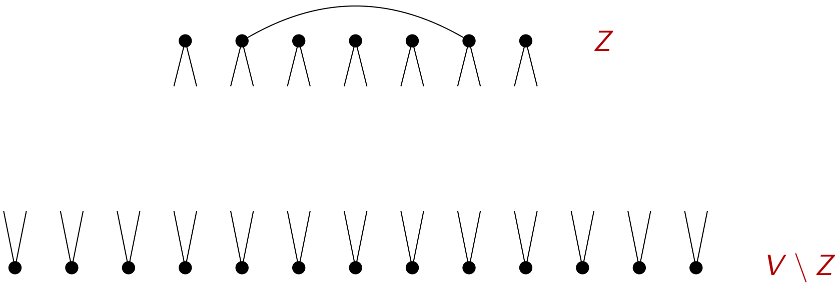
How do we use Z?



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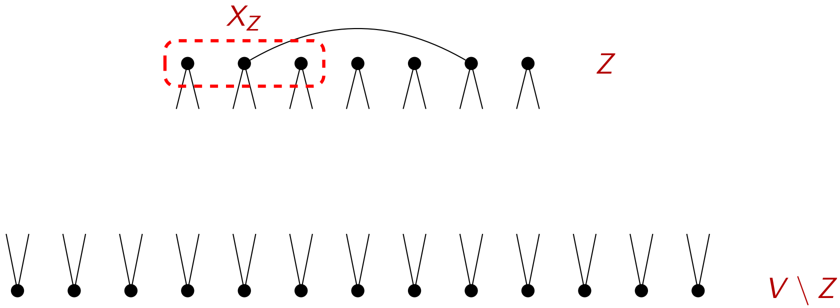
How do we use Z?

- Guess $X \cap Z = X_Z$ (by branching into $2^{|Z|} \leqslant 4^k$ cases).



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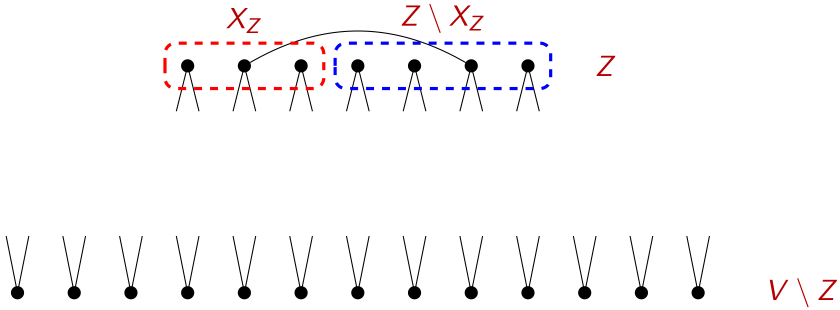

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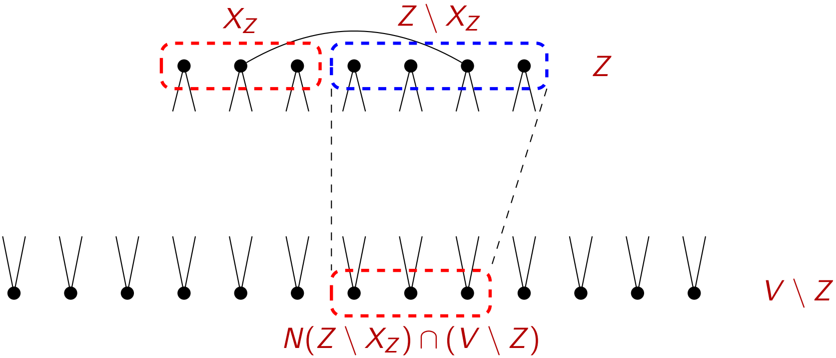

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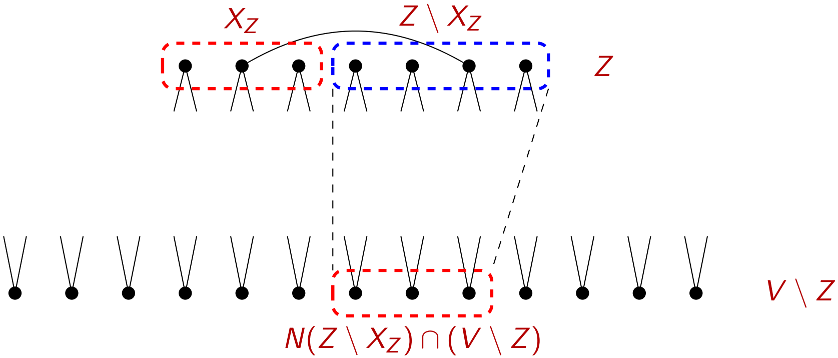
How do we use Z?

- Guess $X \cap Z = X_Z$ (by branching into $2^{|Z|} \leqslant 4^k$ cases).
- Check if $Z \setminus X_Z$ is independent and $|X_Z \cup N(Z \setminus X_Z)| \leqslant k$.



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How do we use Z?

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How do we use Z?

- We have obtained $2^{|Z|}n^{\mathcal{O}(1)} \leqslant 4^k n^{\mathcal{O}(1)}$ time algorithm.
- Can we improve the dependency on k to 2^k?
- Notice that it would be enough to have $|Z| \leqslant k+1$, but so far we only have $|Z| \leqslant 2k$.

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Idea: get Z from recursion!

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Lemma

$f(k)n^c$ time algorithm for VC Compression
implies $f(k)n^{c+1}$ time algorithm for VC.

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Reduction: Vertex Cover $\rightarrow$ Vertex Cover Compression.

Vertex Cover Compression can be solved in time $2^{|Z|}n^{\mathcal{O}(1)}$,
which leads to $2^k n^{\mathcal{O}(1)}$ algorithm for VC.

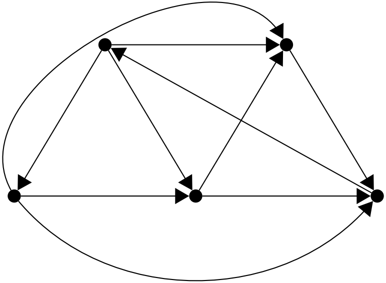
Outline

1. Iterative compression - introduction.
2. Learning by example - vertex cover.
3. Learning by example - FVS in tournament.
4. Generic steps of the method.
5. $5^k n^{\mathcal{O}(1)}$ algorithm for FVS.
6. $3^k n^{\mathcal{O}(1)}$ algorithm for OCT - sketch.

FVS in tournaments

Feedback Vertex Set (FVS) in Tournaments

Input: a tournament (oriented clique) T, integer k
Question: is there a subset $X \subseteq V(T)$ of size at most k,
such that $T \setminus X$ is acyclic



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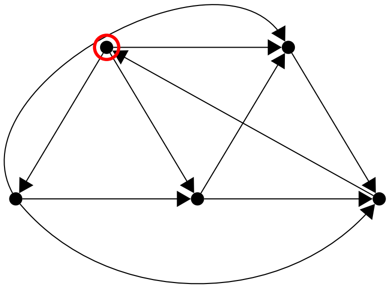
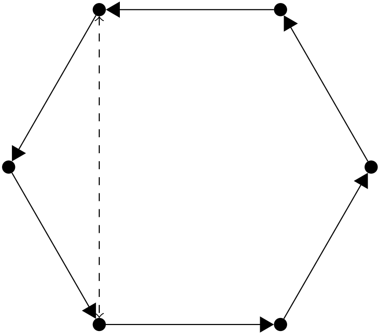


FVS in tournaments

Lemma

If a tournament contains a cycle, then it contains a 3-cycle.



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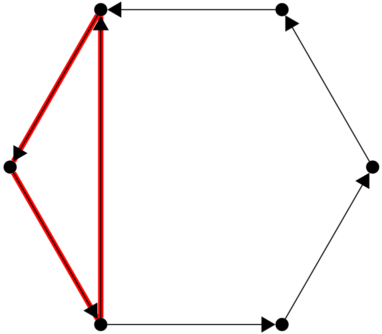

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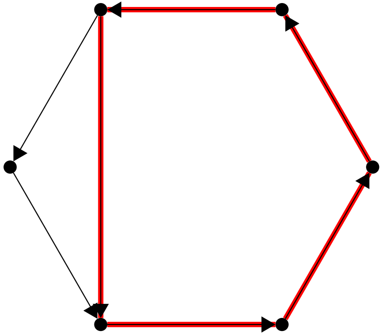

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- This lemma implies a simple $3^k n^{\mathcal{O}(1)}$ branching algorithm.
- By using iterative compression we will see how to improve the running time to $2^k n^{\mathcal{O}(1)}$.

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Lemma

$f(k)n^c$ time algorithm for FVST Compression implies $f(k)n^{c+1}$ time algorithm for FVST.

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Pf: this time we use induction (loop) - alternative to recursion.

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- Let $V(T) = \{v_1, \ldots, v_n\}$.
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- Set $X_1 = \emptyset$, which is a solution for $FVST(T[v_1], k)$.
- For $2 \leqslant i \leqslant n$ do
  - $Z_i = X_{i-1} \cup \{v_i\}$,
  - let X_i be a solution to $FVST\ Compression(T[V_i], k, Z_i)$.
  - if no solution found for $T[V_i]$, then return NO.

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Disjoint FVS in Tournaments Compression

Input: a tournament (oriented clique) T, integer k
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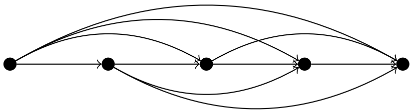
Lemma

Poly time algorithm for Disjoint FVST Compression implies $2^k n^{\mathcal{O}(1)}$ time algorithm for FVST Compression.

Disjoint FVS in tournaments

Observation

For an acyclic tournament, there is a single topological ordering.



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If $T[W]$ is not acyclic, then answer NO.

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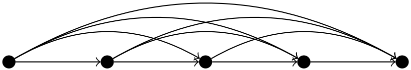
If $T[W]$ is not acyclic, then answer NO.

Let $A = V(T) \setminus W$ (removable set).

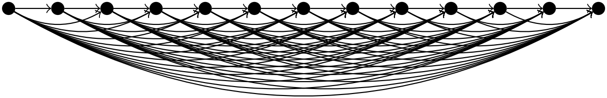
Reduction 2

If for $v \in A$ the graph $T[W \cup \{v\}]$ contains a cycle,
then remove v and reduce k by one.

Disjoint FVS in tournaments

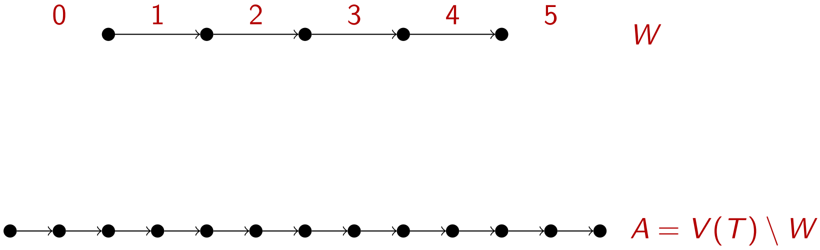


W



$A = V(T) \setminus W$

Disjoint FVS in tournaments



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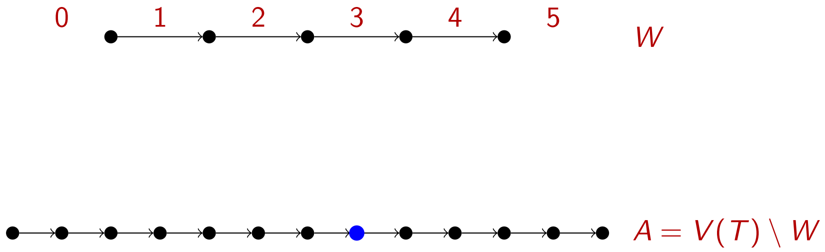

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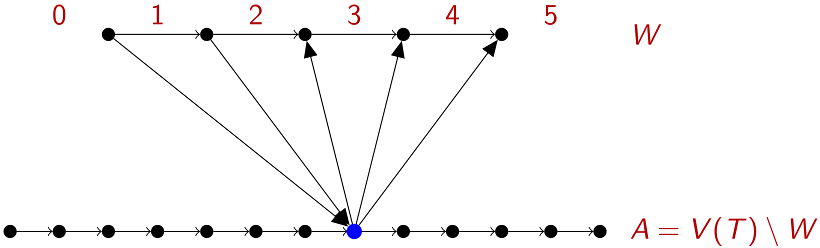

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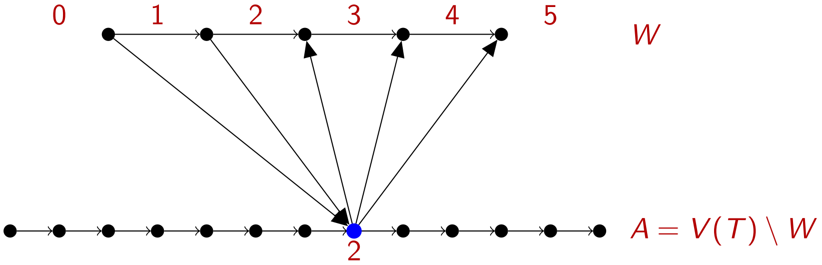

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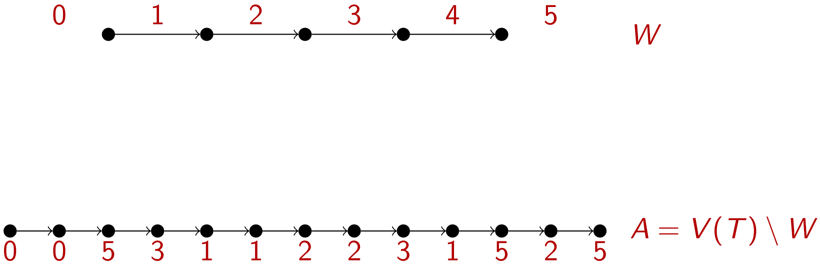

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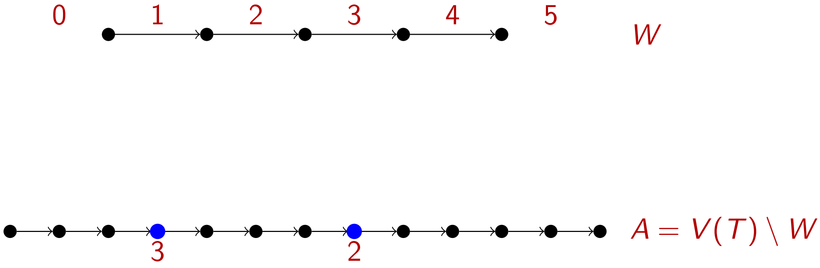

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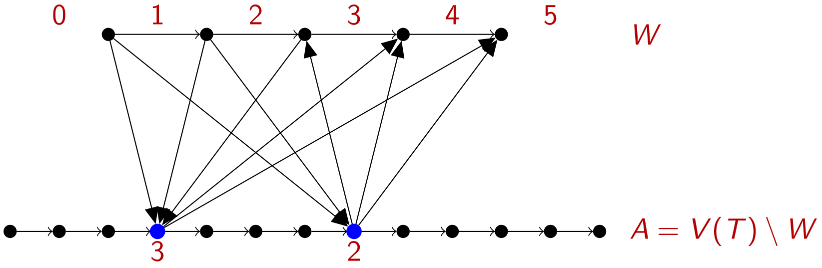

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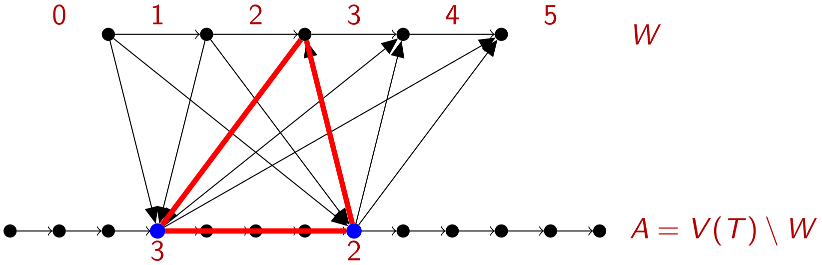

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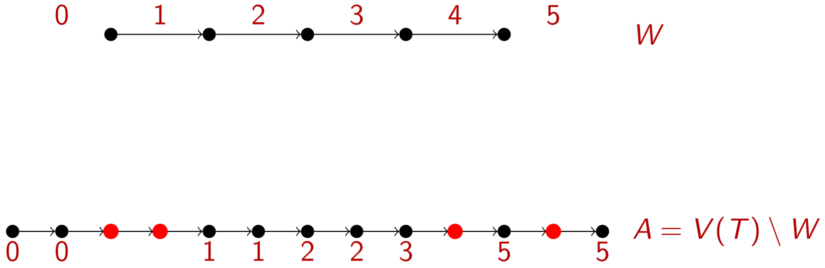

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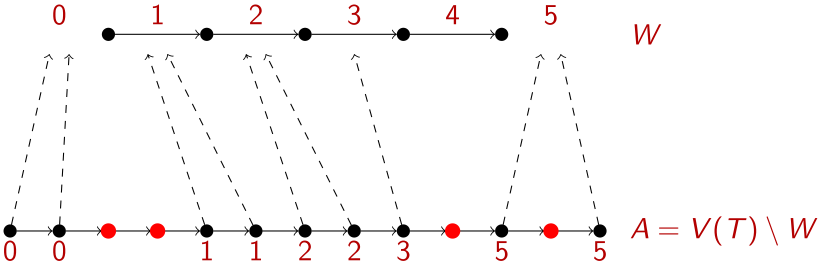

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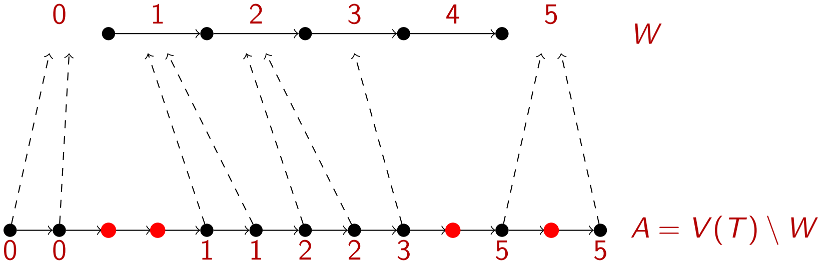

Consequently Disjoint FVST Compression may be reduced to finding longest nondecreasing subsequence.

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Iterative compression schema:

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- $c^k n^{\mathcal{O}(1)}$ time algorithm for the disjoint version implies $(2c)^k n^{\mathcal{O}(1)}$ time algorithm for the general problem.

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$$\sum_{X \subseteq Z} c^{k-|X|} = \sum_{i=0}^{k+1} \binom{k+1}{i} c^{k-i} 1^i = (c+1)^{k+1}/c$$

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- Some natural problems are not vertex deletion closed.
- Ex: reduce Connected Vertex Cover (CVC) to CVC-Compression.

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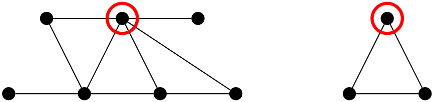
Feedback Vertex Set (FVS)

Input: undirected G, integer k
Question: is there a subset $X \subseteq V(G)$ of size at most k, such that $G \setminus X$ is a forest



FVS

FVS is vertex deletion closed, so we can apply iterative compression schema and solving the following problem in time $c^k n^{\mathcal{O}(1)}$ leads to $(c+1)^k n^{\mathcal{O}(1)}$ time algorithm for FVS.

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Disjoint FVS Compression

Input: undirected G, integer k
a FVS $W \subseteq V(G)$ of size at most $k+1$

Question: is there a subset $X \subseteq V(G)$ of size at most k, disjoint with W, such that $G \setminus X$ is a forest

FVS - reduction rules

Reduction 0

If $G[W]$ contains a cycle, return NO.

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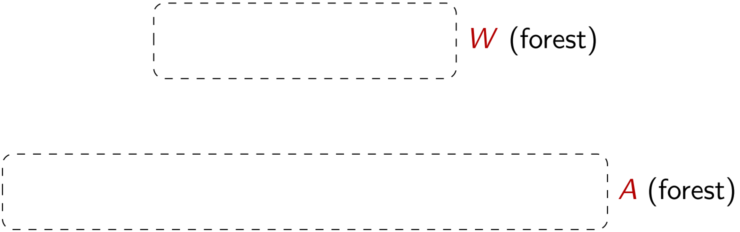

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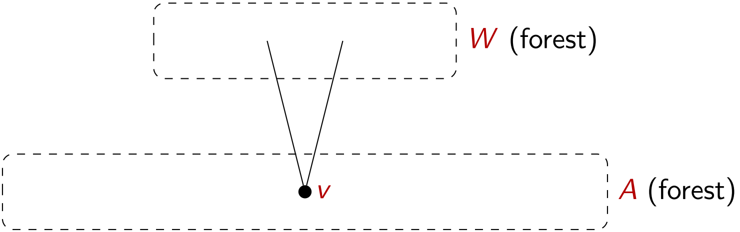
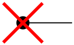


We want v to have $\geqslant 2$ incident edges going to W.

FVS - reduction rules

Reduction 1

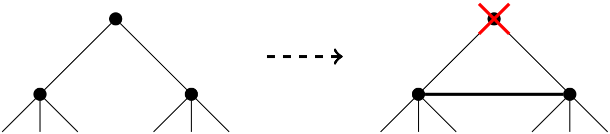
Remove all degree at most 1 vertices from G.



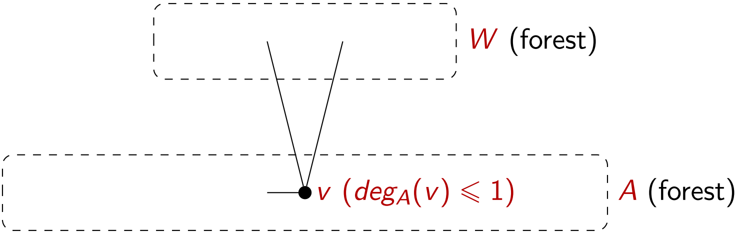
FVS - reduction rules

Reduction 2

If there is $v \in A$ with $\deg(v) = 2$ and at least one neighbor in A, then add an edge between neighbours of v (even if there was one) and remove v.

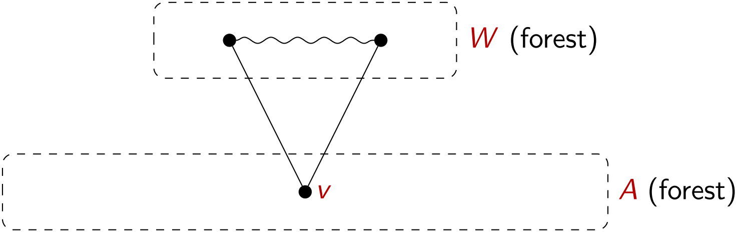


FVS - reduction rules



Any leaf v in A has now at least two edges to W.

FVS - one more reduction rule



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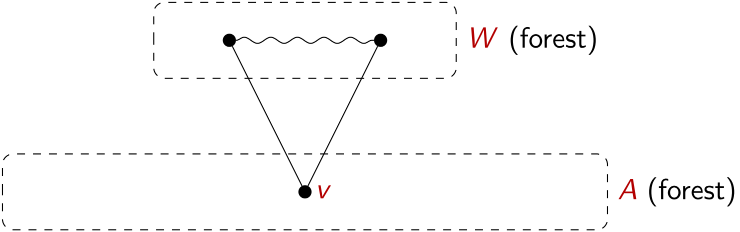
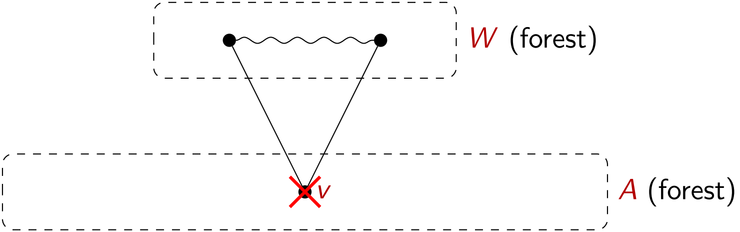


Reduction 3

If for $v \in A = V(G) \setminus W$ the graph $G[W \cup \{v\}]$ contains a cycle, then remove v and decrease k by one.

FVS - one more reduction rule



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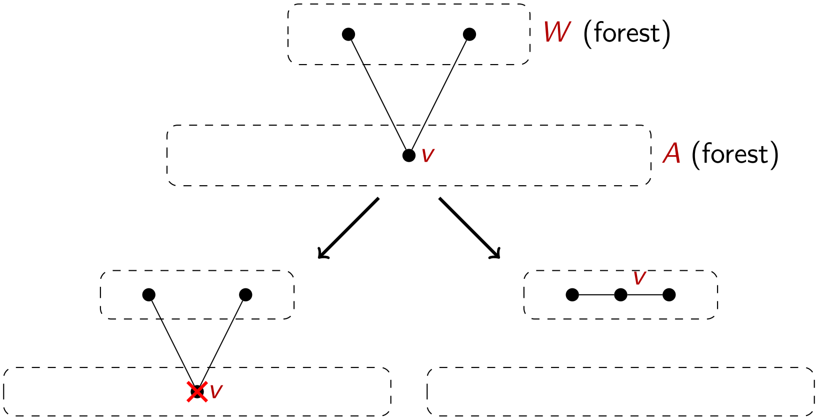

FVS branching

Formally, we branch into instances:

- $(G \setminus \{v\}, k-1, W)$,
- $(G, k, W \cup \{v\})$.

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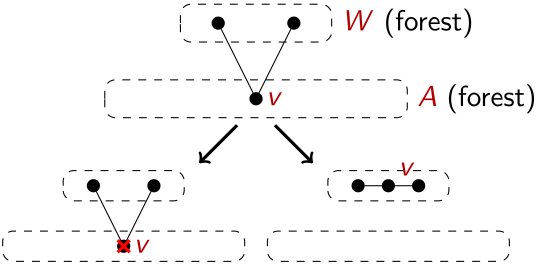
Observation

A potential $\pi(I) = k + \#cc(G[W])$ decreases in each branch.



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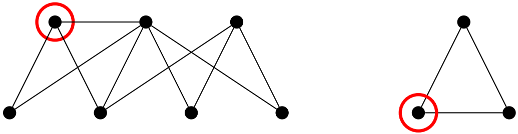
Lemma

Disjoint FVS Compression can be solved in time $4^k n^{\mathcal{O}(1)}$, consequently there is $5^k n^{\mathcal{O}(1)}$ time algorithm for FVS.

OCT

Odd Cycle Transversal (OCT)

Input: undirected G, integer k
Question: is there a subset $X \subseteq V(G)$ of size at most k, such that $G \setminus X$ is bipartite



OCT

The heart of the solution for OCT by iterative compression is the following problem, which can be solved in polynomial time!

Annotated Bipartite Coloring

Input: bipartite $G = (V_1, V_2, E)$, integer k,
a partial coloring $f_0 : V(G) \to \{1, 2, ?\}$

Question: is there a subset $X \subseteq V(G)$ of size at most k,
and a proper coloring f of $G \setminus X$ consistent with f_0.

OCT

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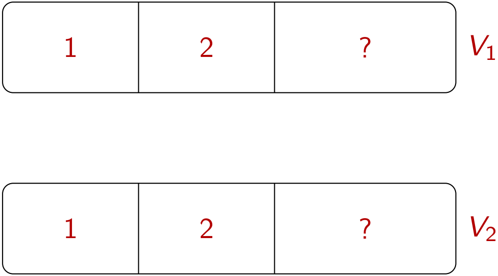

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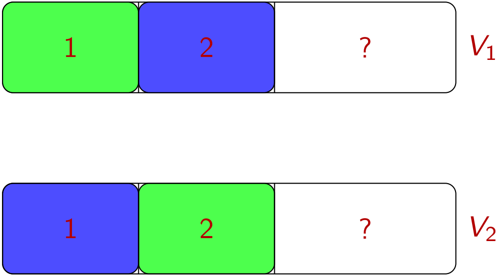

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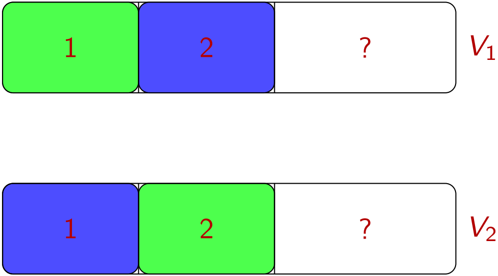
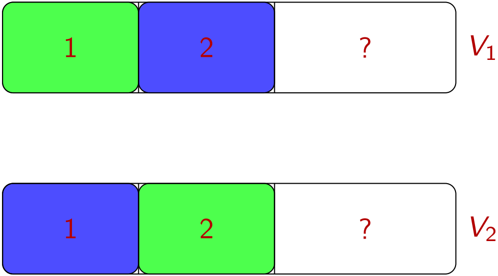


- each blue vertex is either removed or recolored wrt $V_1 \uplus V_2$,

OCT



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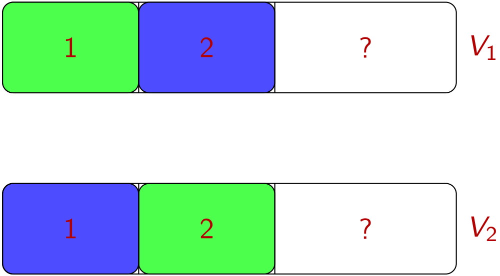
- each green vertex is removed or maintains color wrt $V_1 \uplus V_2$,

OCT



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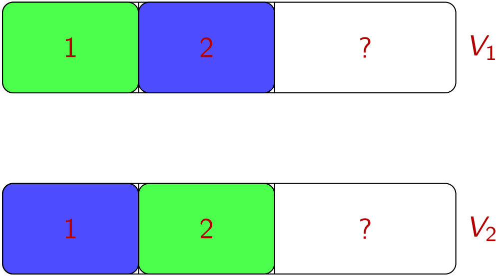

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- each green vertex is removed or maintains color wrt $V_1 \uplus V_2$,
- for each $e \in E(G \setminus X)$ either both vertices are recolored, or none,
- algorithm: find min cut between green and blue!

Summary

Iterative compression

Recursive approach exploiting instance structure exposed by a bit oversized solution.

We have seen it applied to:

- Vertex Cover,
- FVS in Tournaments,
- FVS,
- OCT (sketch).